FOURIER ANALYSIS TECHNIQUES

FOURIER SERIES

Fourier series permit the extension of steady state analysis to general periodic signal.

FOURIER SERIES

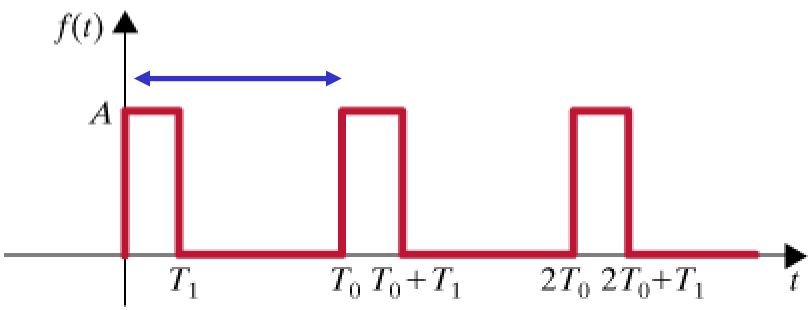
The Fourier series permits the representation of an arbitrary periodic signal as a sum of sinusoids or complex exponentials

Periodic signal

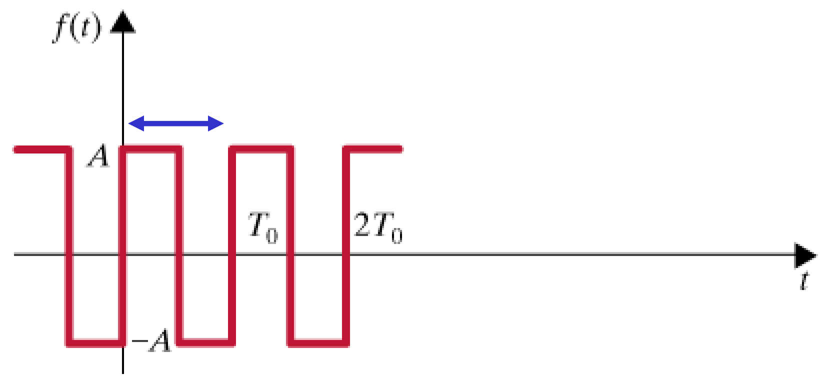
The signal $f(t)$ is periodic iff there exists $T > 0$ such that

$$f(t) = f(t+T), \forall t$$

The *smallest T* that satisfies the previous condition is called the (fundamental) period of the signal



(a)



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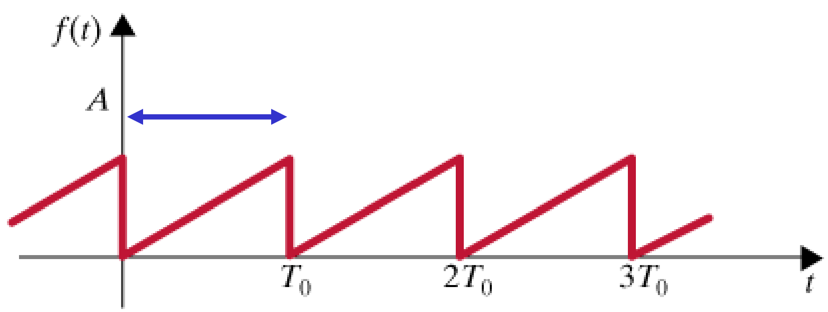

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FOURIER SERIES RESULTS

If $f(t)$ is periodic, with period T_0, then $f(t)$ can be expressed in one of the following equivalent forms

$$\omega_0 = \frac{2\pi}{T_0}$$

Cosine expansion

Phasor for n-th harmonic

$$f(t) = a_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_o t + \theta_n) = a_0 + \sum_{n=1}^{\infty} \mathrm{Re}\left[D_n \angle \theta_n e^{jn\omega_o t}\right]$$

$$D_n \angle \theta_n = 2c_n = a_n - jb_n$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Complex exponential expansion

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$

$$e^{j\alpha} + e^{-j\alpha} = 2\cos\alpha$$

$$e^{j\alpha} - e^{-j\alpha} = 2j\sin\alpha$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_o t + \sum_{n=1}^{\infty} b_n \sin n\omega_o t$$

Trigonometric series

Relationship between exponential and trigonometric expansions

$c_0 = a_0$

For $n > 0$

$$c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} = \underbrace{(c_n + c_{-n})}_{a_n} \cos n\omega_0 t + \underbrace{j(c_n - c_{-n})}_{b_n} \sin n\omega_0 t$$

$$\Rightarrow 2c_n = a_n - jb_n$$

$$2c_{-n} = a_n + jb_n$$

If $f(t)$ is real-valued then $c_{-n} = (c_n)^*$

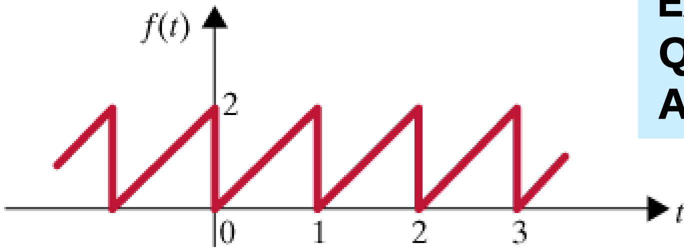
GENERAL STRATEGY:

. Approximate a periodic signal using a Fourier series

. Analyze the network for each harmonic using phasors or complex exponentials

. Use the superposition principle to determine the response to the periodic signal

Original Periodic Signal

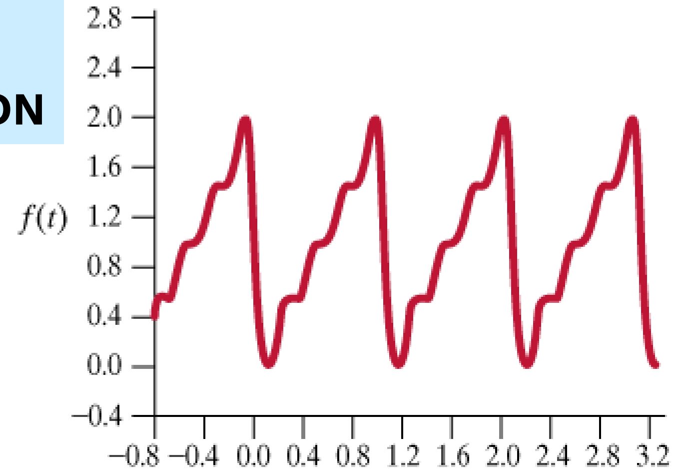


(a)

EXAMPLE OF QUALITY OF APPROXIMATION

$$f_N(t) = \sum_{n=-N}^{N} c_n e^{jn\omega_0 t}$$

Approximation with 4 terms



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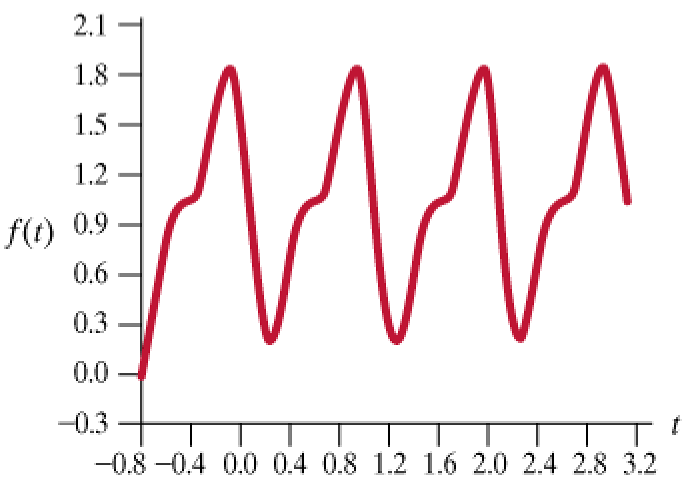
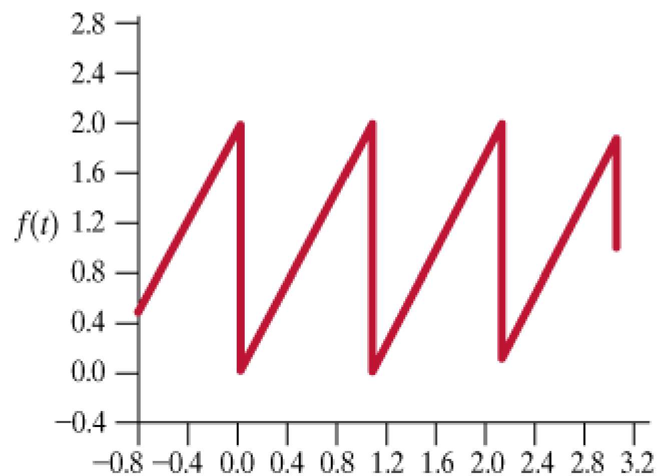
Approximation with 2 terms



$$f_2(t) = a_0 + \sum_{n=1}^{2} a_n \cos n\omega_o t + \sum_{n=1}^{2} b_n \sin n\omega_o t$$

Approximation with 100 terms



$$a_0 + \sum_{n=1}^{100} a_n \cos n\omega_o t + \sum_{n=1}^{100} b_n \sin n\omega_o t$$

EXPONENTIAL FOURIER SERIES

Any "physicallv realizable" periodic signal, with period To, can be represented over the interval $t_1 < t < t_1 + T_0$ by the expression

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T_0}$$

The sum of exponential functions is always a continuous function. Hence, the right hand side is a continuous function.
Technically, one requires the signal, *f(t),* to be at least piecewise continuous. In that case, the equality does not hold at the points where the signal is discontinuous

Computation of the exponential Fourier series coefficients

$$\int_{t_1}^{t_1+T_0} \qquad f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} \qquad \times e^{-jk\omega_0 t}$$

$$\int_{t_1}^{t_1+T_0} e^{j(n-k)\omega_0 t} dt = \begin{cases} 0 & \text{for } n \neq k \\ T_0 & \text{for } n = k \end{cases}$$

$$\int_{t_1}^{t_1+T_0} f(t) e^{-jk\omega_0 t} dt = \int_{t_1}^{t_1+T_0} \left(\sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} \right) e^{-jk\omega_0 t} dt$$

$$c_k = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) e^{-jk\omega_0 t} dt$$

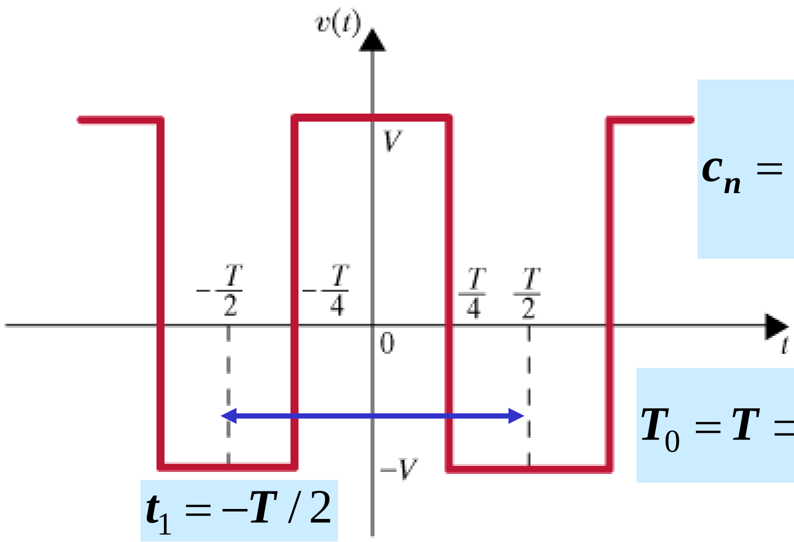
t_1 is arbitray and can be chosen to make computations simpler

EXAMPLE

Determine the exponential Fourier series

$$f(t)=\sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}; \quad \omega_0=\frac{2\pi}{T_0}$$

$$c_n=\frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t)e^{-jn\omega_0 t}dt$$



$$c_n=\begin{cases} 0 & n \quad even \\ \frac{2V}{\pi n}\sin\frac{n\pi}{2} & n \quad odd \end{cases}$$

$$t_1=-T/2$$

$$T_0=T \Rightarrow \omega_0=\frac{2\pi}{T}$$

A strategy:

1. Determine T_0 and ω_0
2. Select a convenient t_1
3. Do the integration

$$c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} v(t)e^{-jn\omega_0 t}dt=\frac{1}{T}\int_{-\frac{T}{2}}^{-\frac{T}{4}}(-V)e^{-jn\omega_0 t}dt+\frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} Ve^{-jn\omega_0 t}dt-\frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} Ve^{-jn\omega_0 t}dt$$

$$\frac{e^{j\alpha}-e^{-j\alpha}}{j}=2\sin\alpha$$

$$T\omega_0=2\pi$$

$$c_n=\frac{V}{T(jn\omega_0)}\left[e^{-jn\omega_0 t}\Big|_{-T/2}^{-T/4}-e^{-jn\omega_0 t}\Big|_{-T/4}^{T/4}+e^{-jn\omega_0 t}\Big|_{T/4}^{T/2}\right]$$

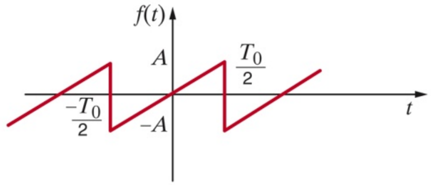
This is for $n \neq 0$!

$c_0=0$ in this case

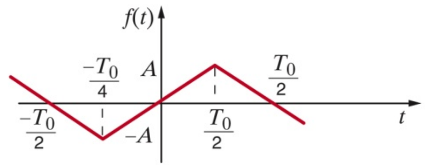
$$c_n=\frac{V}{jTn\omega_0}\left[e^{-jn\omega_0\left(-\frac{T}{4}\right)}-e^{-jn\omega_0\left(-\frac{T}{2}\right)}-e^{-jn\omega_0\left(\frac{T}{4}\right)}+e^{-jn\omega_0\left(-\frac{T}{4}\right)}+e^{-jn\omega_0\left(\frac{T}{2}\right)}-e^{-jn\omega_0\left(\frac{T}{4}\right)}\right]$$

$$c_n=\frac{V}{j2\pi n}\left[2e^{j\frac{n\pi}{2}}-2e^{-j\frac{n\pi}{2}}+e^{-jn\pi}-e^{jn\pi}\right]=\frac{V}{2\pi n}\left[4\sin\left(\frac{n\pi}{2}\right)-2\sin(n\pi)\right]$$

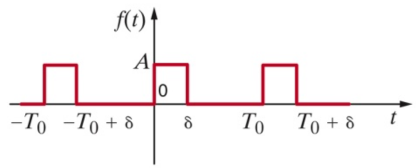
TABLE 15.2 Fourier series for some common waveforms



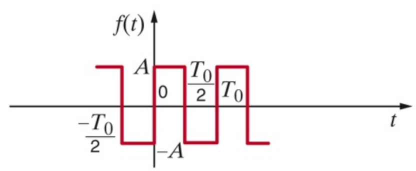
$$f(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2A}{n\pi} \sin n\omega_o t$$



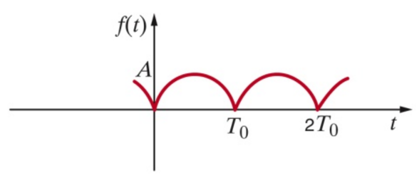
$$f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8A}{n^2\pi^2} \sin \frac{n\pi}{2} \sin n\omega_o t$$



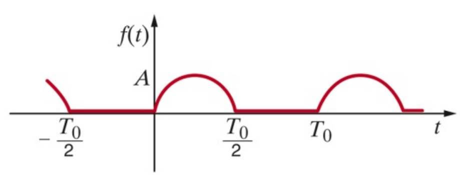
$$f(t) = \sum_{n=-\infty}^{\infty} \frac{A}{n\pi} \sin \frac{n\pi\delta}{T_o} e^{jn\omega_o[t-(\delta/2)]}$$



$$f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4A}{n\pi} \sin n\omega_o t$$



$$f(t) = \frac{2A}{\pi} + \sum_{n=1}^{\infty} \frac{4A}{\pi(1 - 4n^2)} \cos n\omega_o t$$



$$f(t) = \frac{A}{\pi} + \frac{A}{2} \sin \omega_o t + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{2A}{\pi(1 - n^2)} \cos n\omega_o t$$

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