FIRST ORDER TRANSIENT CIRCUITS

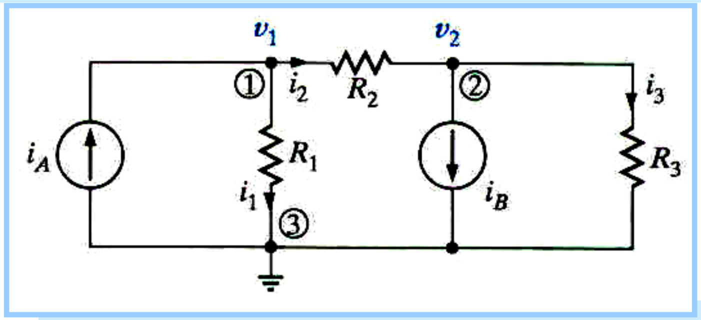
IN CIRCUITS WITH INDUCTORS AND CAPACITORS VOLTAGES AND CURRENTS CANNOT CHANGE INSTANTANEOUSLY.
EVEN THE APPLICATION, OR REMOVAL, OF CONSTANT SOURCES CREATES A TRANSIENT BEHAVIOR

FIRST ORDER CIRCUITS
Circuits that contain a single energy storing elements.
Either a capacitor or an inductor

ANALYSIS OF LINEAR CIRCUITS WITH INDUCTORS AND/OR CAPACITORS

THE CONVENTIONAL ANALYSIS USING MATHEMATICAL MODELS REQUIRES THE DETERMINATION OF (A SET OF) EQUATIONS THAT REPRESENT THE CIRCUIT.
ONCE THE MODEL IS OBTAINED ANALYSIS REQUIRES THE SOLUTION OF THE EQUATIONS FOR THE CASES REQUIRED.

FOR EXAMPLE IN NODE OR LOOP ANALYSIS OF RESISTIVE CIRCUITS ONE REPRESENTS THE CIRCUIT BY A SET OF ALGEBRAIC EQUATIONS



THE MODEL

$$(G_1 + G_2)v_1 - G_2 v_2 = i_A$$
$$-G_2 v_1 + (G_2 + G_3)v_2 = -i_B$$

WHEN THERE ARE INDUCTORS OR CAPACITORS THE MODELS BECOME LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODEs). HENCE, IN GENERAL, ONE NEEDS ALL THOSE TOOLS IN ORDER TO BE ABLE TO ANALYZE CIRCUITS WITH ENERGY STORING ELEMENTS.

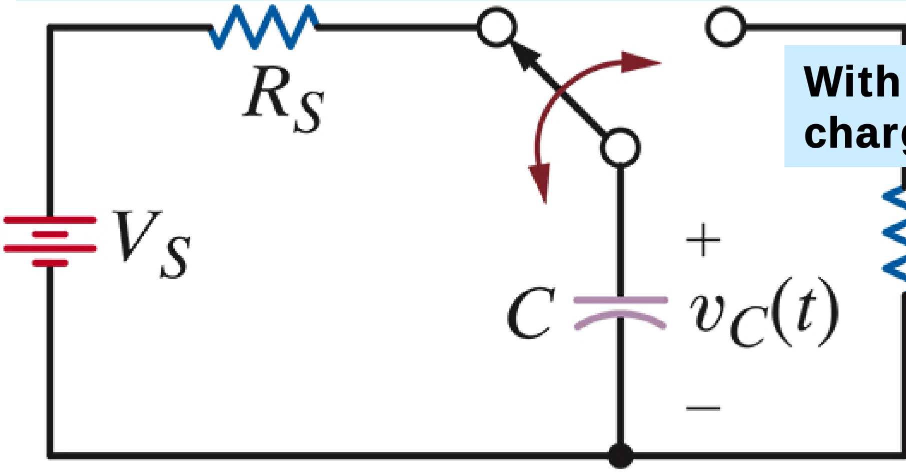
A METHOD BASED ON THEVENIN WILL BE DEVELOPED TO DERIVE MATHEMATICAL MODELS FOR ANY ARBITRARY LINEAR CIRCUIT WITH ONE ENERGY STORING ELEMENT.

THE GENERAL APPROACH CAN BE SIMPLIFIED IN SOME SPECIAL CASES WHEN THE FORM OF THE SOLUTION CAN BE KNOWN BEFOREHAND.
THE ANALYSIS IN THESE CASES BECOMES A SIMPLE MATTER OF DETERMINING SOME PARAMETERS.
TWO SUCH CASES WILL BE DISCUSSED IN DETAIL FOR THE CASE OF CONSTANT SOURCES.
ONE THAT ASSUMES THE AVAILABILITY OF THE DIFFERENTIAL EQUATION AND A SECOND THAT IS ENTIRELY BASED ON ELEMENTARY CIRCUIT ANALYSIS… BUT IT IS NORMALLY LONGER

WE WILL ALSO DISCUSS THE PERFORMANCE OF LINEAR CIRCUITS TO OTHER SIMPLE INPUTS

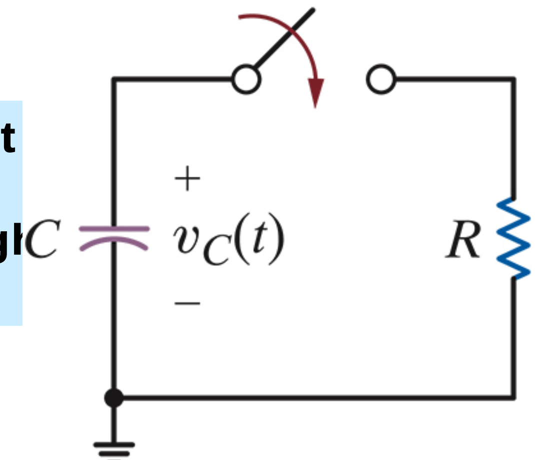
AN INTRODUCTION

INDUCTORS AND CAPACITORS CAN STORE ENERGY. UNDER SUITABLE CONDITIONS THIS ENERGY CAN BE RELEASED. THE RATE AT WHICH IT IS RELEASED WILL DEPEND ON THE PARAMETERS OF THE CIRCUIT CONNECTED TO THE TERMINALS OF THE ENERGY STORING ELEMENT

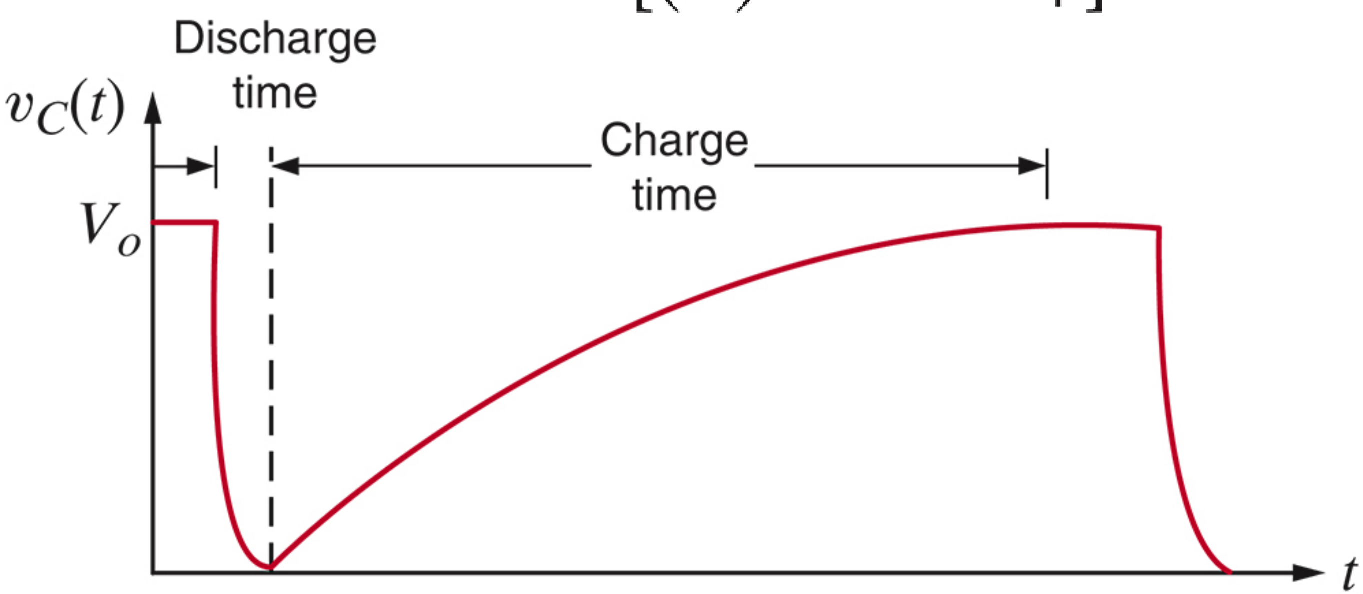


With the switch on the left the capacitor receives charge from the battery.

Switch to the right and the capacitor discharges through the lamp



[(R) Xenon lamp]



GENERAL RESPONSE: FIRST ORDER CIRCUITS

Including the initial conditions the model for the capacitor voltage or the inductor current will be shown to be of the form

$$\frac{dx}{dt}(t) + ax(t) = f(t);\ \ x(0+) = x_0$$

$$\tau\frac{dx}{dt} + x = f_{TH};\ \ x(0+) = x_0$$

Solving the differential equation using integrating factors, one tries to convert the LHS into an exact derivative

$$\tau\frac{dx}{dt} + x = f_{TH} \quad /*\frac{1}{\tau}e^{\frac{t}{\tau}}$$

$$e^{\frac{t}{\tau}}\frac{dx}{dt} + \frac{1}{\tau}e^{\frac{t}{\tau}}x = \frac{1}{\tau}e^{\frac{t}{\tau}}f_{TH}$$

$$\int_{t_0}^{t} \quad \frac{d}{dt}\left(e^{\frac{t}{\tau}}x\right) = \frac{1}{\tau}e^{\frac{t}{\tau}}f_{TH}$$

$$e^{\frac{t}{\tau}}x(t) - e^{\frac{t_0}{\tau}}x(t_0) = \int_{t_0}^{t}\frac{1}{\tau}e^{\frac{x}{\tau}}f_{TH}(x)dx \quad */e^{-\frac{t}{\tau}}$$

$$x(t) = e^{-\frac{t-t_0}{\tau}}x(t_0) + \frac{1}{\tau}\int_{t_0}^{t}e^{-\frac{t-x}{\tau}}f_{TH}(x)dx$$

THIS EXPRESSION ALLOWS THE COMPUTATION OF THE RESPONSE FOR ANY FORCING FUNCTION. WE WILL CONCENTRATE IN THE SPECIAL CASE WHEN THE RIGHT HAND SIDE IS CONSTANT

τ is called the "time constant."
it will be shown to provide significant information on the reaction speed of the circuit

The initial time, t_o, is arbitrary. The general expression can be used to study sequential switchings.

FIRST ORDER CIRCUITS WITH CONSTANT SOURCES

$$\tau \frac{dx}{dt} + x = f_{TH}; \quad x(0+) = x_0$$

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + \frac{1}{\tau}\int_{t_0}^{t} e^{-\frac{t-x}{\tau}} f_{TH}(x)dx$$

If the RHS is constant

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + \frac{f_{TH}}{\tau}\int_{t_0}^{t} e^{-\frac{t-x}{\tau}} dx$$

$$e^{-\frac{t-x}{\tau}} = e^{-\frac{t}{\tau}} e^{\frac{x}{\tau}} \Rightarrow$$

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + \frac{f_{TH}}{\tau} e^{-\frac{t}{\tau}} \int_{t_0}^{t} e^{\frac{x}{\tau}} dx$$

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + \frac{f_{TH}}{\tau} e^{-\frac{t}{\tau}} \left(\tau e^{\frac{x}{\tau}} \right)_{t_0}^{t}$$

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + f_{TH} e^{-\frac{t}{\tau}} \left(e^{\frac{t}{\tau}} - e^{\frac{t_0}{\tau}} \right)$$

$$x(t) = f_{TH} + \left(x(t_0) - f_{TH} \right) e^{-\frac{t-t_0}{\tau}}$$

$$t \ge t_0$$

The form of the solution is

$$x(t) = K_1 + K_2 e^{-\frac{t-t_0}{\tau}}; \; t \ge t_0$$

TIME CONSTANT

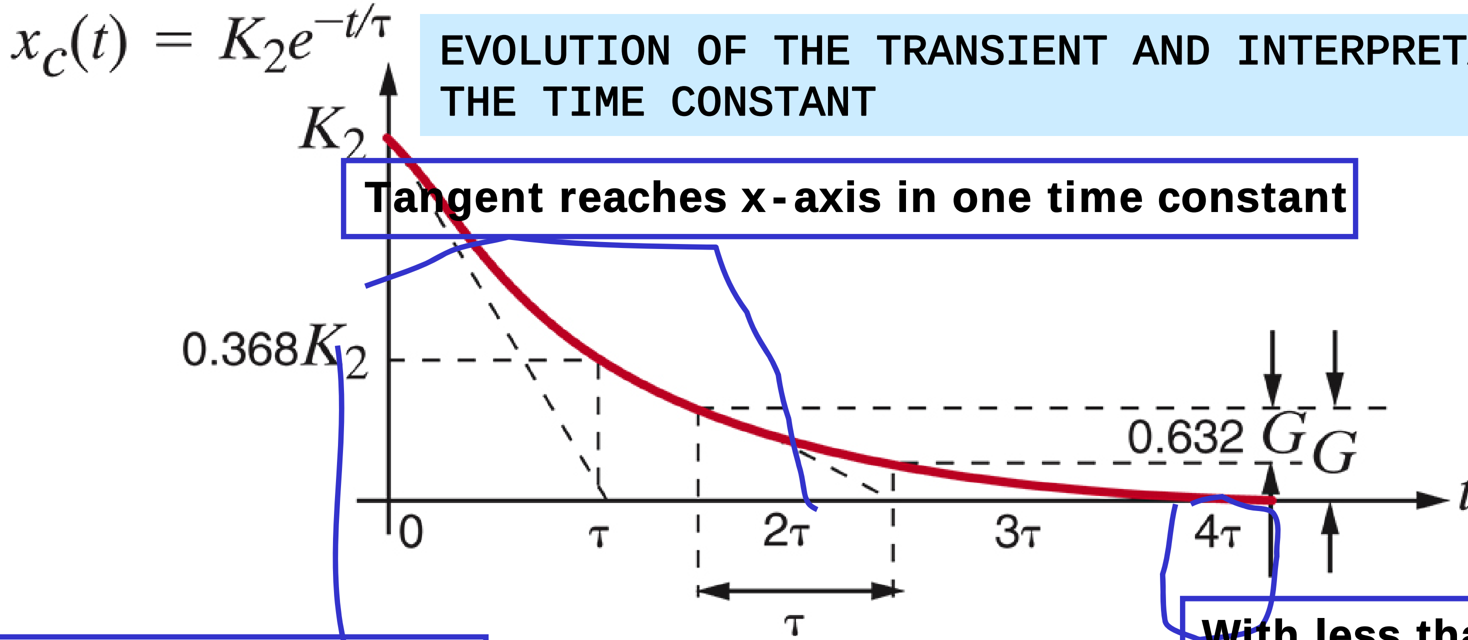
TRANSIENT

Any variable in the circuit is of the form

$$y(t) = K_1 + K_2 e^{-\frac{t-t_0}{\tau}}; \; t \ge t_0$$

Only the values of the constants K_1, K_2 will change

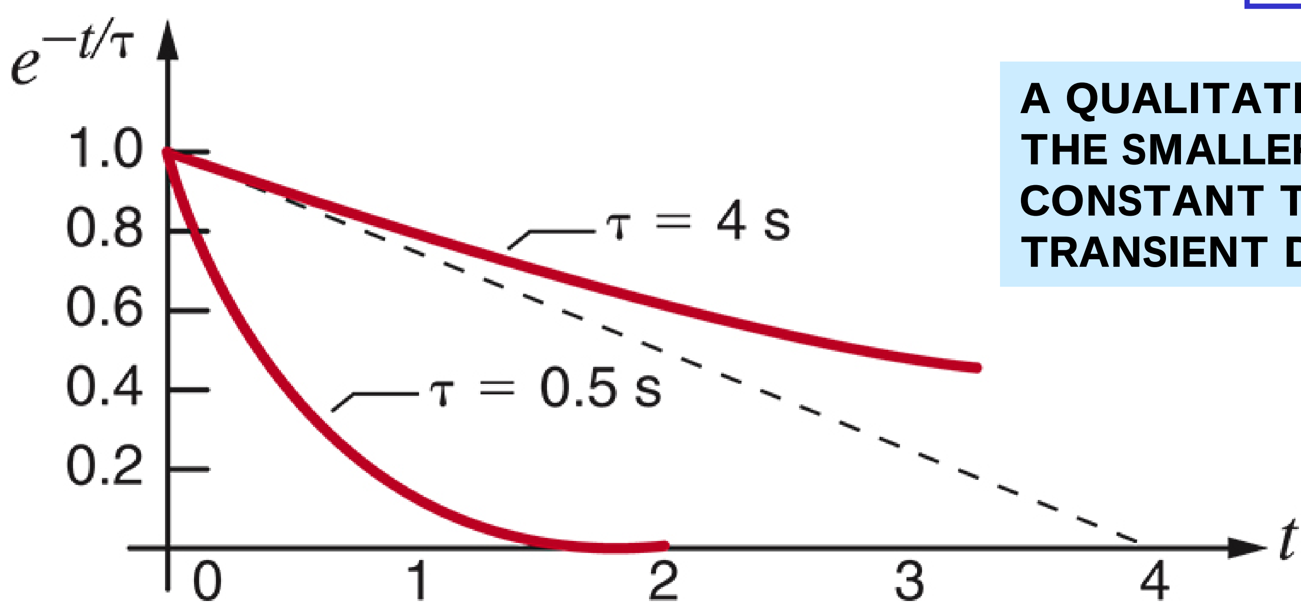
EVOLUTION OF THE TRANSIENT AND INTERPRETATION OF THE TIME CONSTANT



Tangent reaches x-axis in one time constant

Drops 0.632 of initial value in one time constant

With less than 2% error transient is zero beyond this point

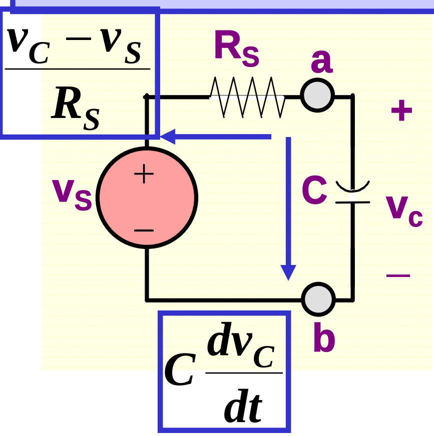


A QUALITATIVE VIEW:
THE SMALLER THE THE TIME CONSTANT THE FASTER THE TRANSIENT DISAPPEARS

THE TIME CONSTANT

The following example illustrates the physical meaning of time constant

Charging a capacitor



KCL@a:

$$C\frac{dv_c}{dt} + \frac{v_C - v_S}{R_S} = 0$$

The model

$$R_{TH}C\frac{dv_C}{dt} + v_C = v_{TH}$$

$$\tau = R_{TH}C$$

Assume

$$v_S = V_S, v_C(0) = 0$$

The solution can be shown to be

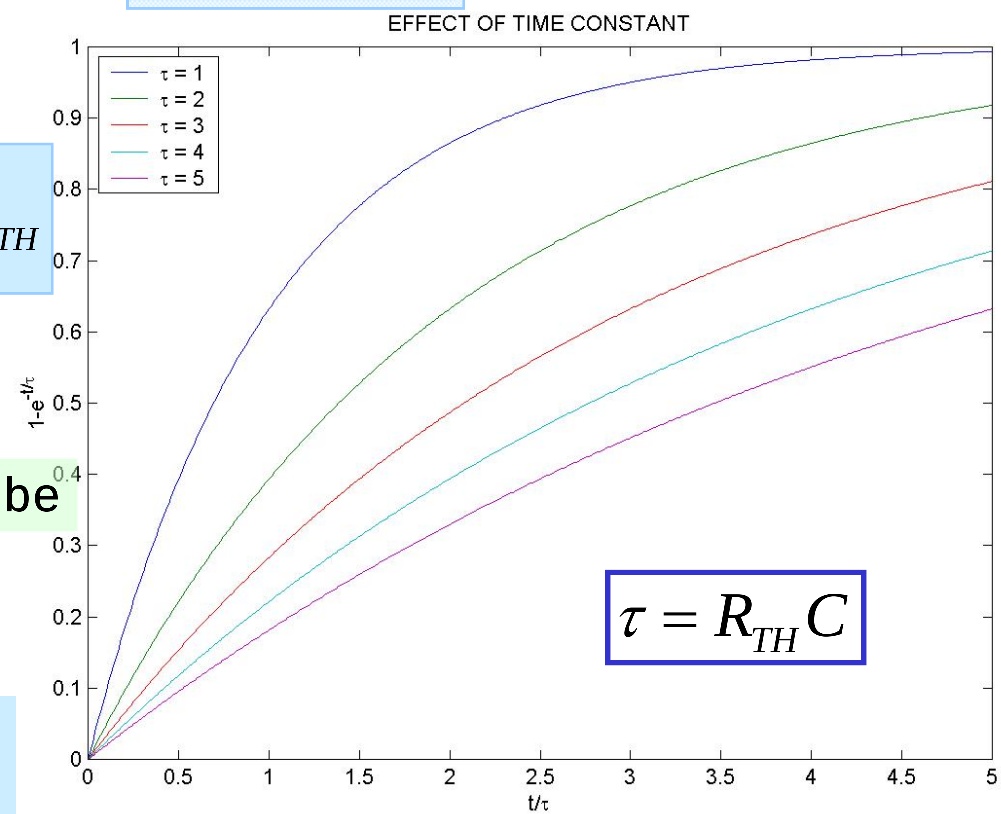
$$v_C(t) = V_S - V_S e^{-\frac{t}{\tau}}$$

transient

For practical purposes the capacitor is charged when the transient is negligible

t	$e^{-\frac{t}{\tau}}$
τ	0.368
2τ	0.135
3τ	0.0498
4τ	0.0183
5τ	0.0067

With less than 1% error the transient is negligible after five time constants



$$\tau = R_{TH}C$$

CIRCUITS WITH ONE ENERGY STORING ELEMENT

THE DIFFERENTIAL EQUATION APPROACH

CONDITIONS

1. THE CIRCUIT HAS ONLY CONSTANT INDEPENDENT SOURCES
2. THE DIFFERENTIAL EQUATION FOR THE VARIABLE OF INTEREST IS SIMPLE TO OBTAIN. NORMALLY USING BASIC ANALYSIS TOOLS; e.g., KCL, KVL. . . OR THEVENIN
3. THE INITIAL CONDITION FOR THE DIFFERENTIAL EQUATION IS KNOWN, OR CAN BE OBTAINED USING STEADY STATE ANALYSIS

FACT: WHEN ALL INDEPENDENT SOURCES ARE CONSTANT FOR ANY VARIABLE, $y(t)$, IN THE CIRCUIT THE SOLUTION IS OF THE FORM

$$y(t) = K_1 + K_2 e^{-\frac{(t-t_O)}{\tau}}, \; t > t_O$$

SOLUTION STRATEGY: USE THE DIFFERENTIAL EQUATION AND THE INITIAL CONDITIONS TO FIND THE PARAMETERS K_1, K_2, τ

If the diff eq for y is known in the form

$$a_1 \frac{dy}{dt} + a_0 y = f$$
$$y(0+) = y_0$$

We can use this info to find the unknowns

Use the diff eq to find two more equations by replacing the form of solution into the differential equation

$$y(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0 \Rightarrow \frac{dy}{dt} = -\frac{K_2}{\tau} e^{-\frac{t}{\tau}}$$

$$a_1\left(-\frac{K_2}{\tau} e^{-\frac{t}{\tau}}\right) + a_0\left(K_1 + K_2 e^{-\frac{t}{\tau}}\right) = f$$

$$a_0 K_1 = f \Rightarrow K_1 = \frac{f}{a_0}$$

$$\left(-\frac{a_1}{\tau} + a_0\right) K_2 e^{-\frac{t}{\tau}} = 0 \Rightarrow \tau = \frac{a_1}{a_0}$$

Use the initial condition to get one more equation

$$y(0+) = K_1 + K_2$$

$$K_2 = y(0+) - K_1$$

SHORTCUT: WRITE DIFFERENTIAL EQ. IN NORMALIZED FORM WITH COEFFICIENT OF VARIABLE = 1.

$$a_1 \frac{dy}{dt} + a_0 y = f \Rightarrow \frac{a_1}{a_0} \frac{dy}{dt} + y = \frac{f}{a_0}$$

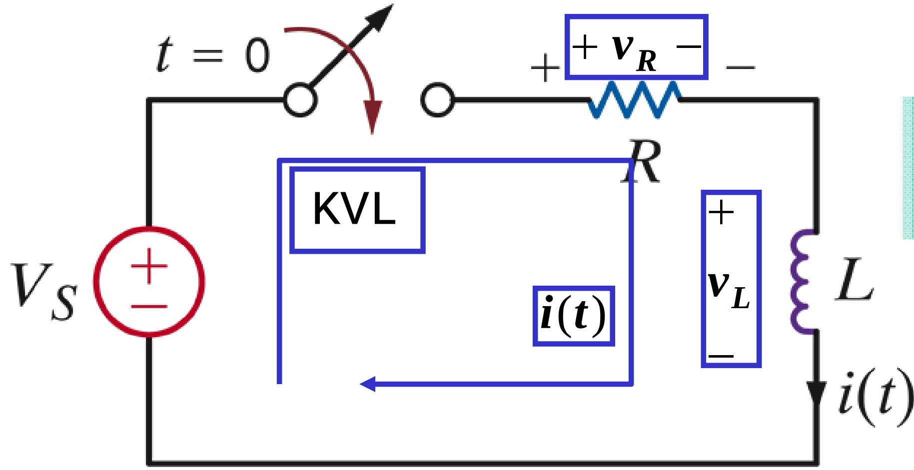
$$\tau \qquad K_1$$

EXAMPLE

FIND $i(t), t>0$

$$x(t)=K_1+K_2e^{-\frac{t}{\tau}}, t>0$$

$$K_1=x(\infty);\ K_1+K_2=x(0+)$$



$$i(t)=K_1+K_2e^{-\frac{t}{\tau}}, t>0$$

MODEL. USE KVL FOR t > 0

$$V_S=v_R+v_L=Ri(t)+L\frac{di}{dt}(t)$$

INITIAL CONDITION

$$\left.\begin{array}{l} t<0 \Rightarrow i(0-)=0 \\ \text{inductor} \Rightarrow i(0-)=i(0+) \end{array}\right\} i(0+)=0$$

STEP 1 $\frac{L}{R}\frac{di}{dt}(t)+i(t)=\frac{V_S}{R}$ $\quad \tau=\frac{L}{R}$

STEP 2 STEADY STATE $i(\infty)=K_1=\frac{V_S}{R}$

STEP 3 INITIAL CONDITION

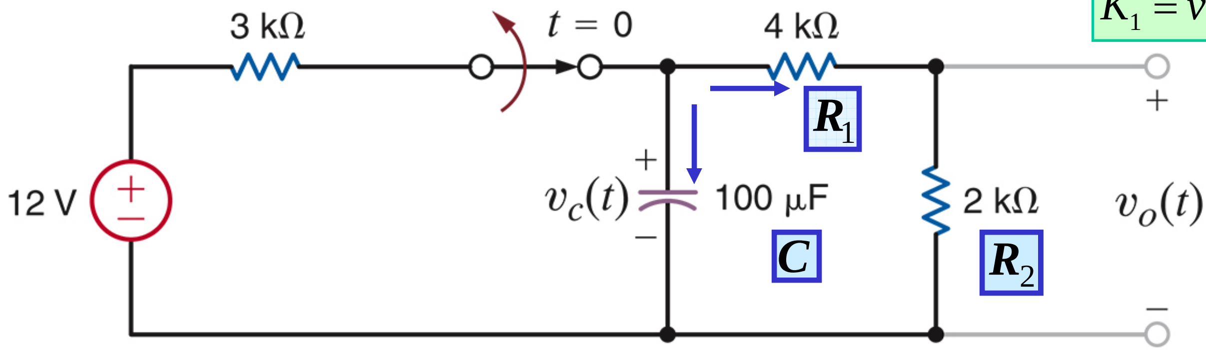
$$i(0+)=K_1+K_2$$

ANS: $i(t)=\frac{V_S}{R}\left(1-e^{-\frac{t}{L/R}}\right)$

FIND $v_O(t), t > 0$

$$v_C(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$

$$K_1 = v_C(\infty);\ K_1 + K_2 = v_C(0+)$$



MODEL FOR $t > 0$. USE KCL

DETERMINE $v_c(t)$

$$C\frac{dv_C}{dt}(t) + \frac{v_C}{R_1 + R_2} = 0 \Rightarrow (R_1 + R_2)C\frac{dv_C}{dt}(t) + v_c = 0$$

$$v_O(t) = \frac{2}{2+4} v_C(t) = \frac{1}{3} v_C(t)$$

STEP 1

$$\tau = (R_1 + R_2)C = (6 \times 10^3 \Omega)(100 \times 10^{-6} F) = 0.6s$$

$$v_O(t) = \frac{8}{3} e^{-\frac{t}{0.6}} [V], t > 0$$

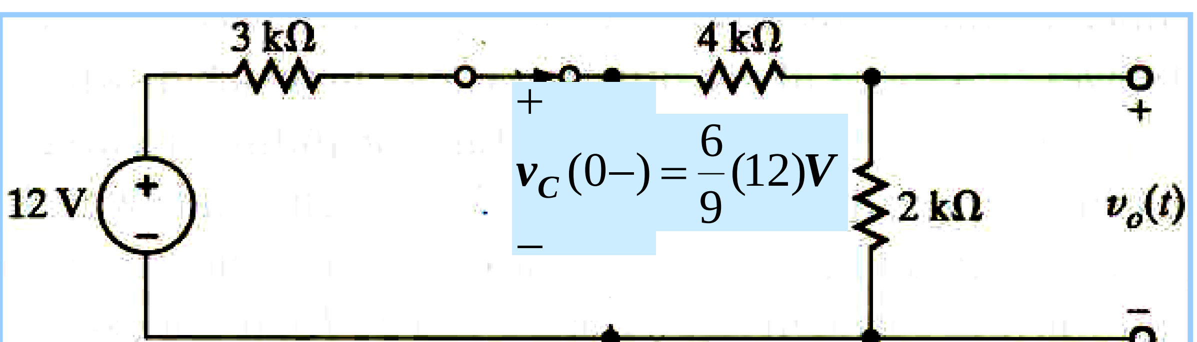
STEP 2

$$v_C(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$

$K_1 = 0$

INITIAL CONDITIONS. CIRCUIT IN STEADY STATE t<0

STEP 3



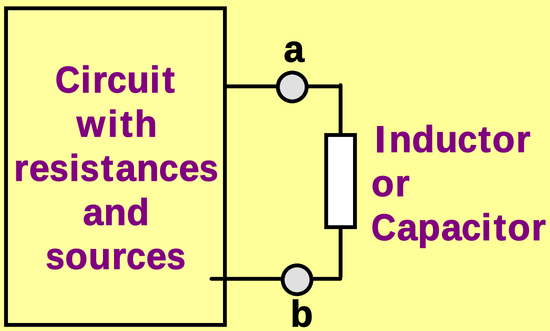
$$v_C(0-) = \frac{6}{9}(12)V$$

$$v_C(0+) = 8 = K_1 + K_2 \Rightarrow K_2 = 8[V]$$

$$v_C(t) = 8e^{-\frac{t}{0.6}} [V], t > 0$$

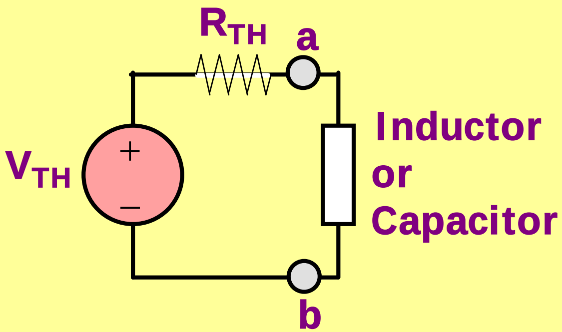
USING THEVENIN TO OBTAIN MODELS

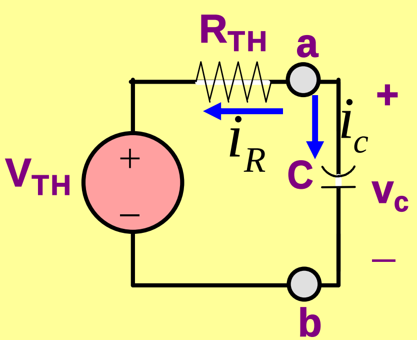
Obtain the voltage across the capacitor or the current through the inductor



Representation of an arbitrary circuit with one storage element

Thevenin





Case 1.1
Voltage across capacitor

KCL@ node a

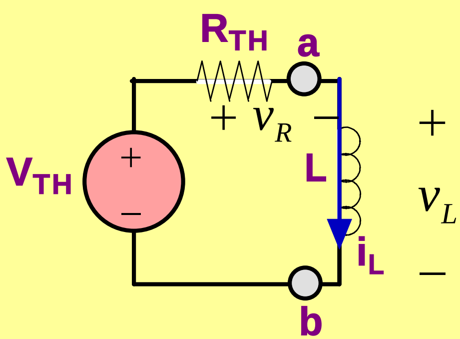
$$i_c + i_R = 0$$

$$i_c = C\frac{dv_C}{dt}$$

$$i_R = \frac{v_C - v_{TH}}{R_{TH}}$$

$$C\frac{dv_C}{dt} + \frac{v_C - v_{TH}}{R_{TH}} = 0$$

$$R_{TH}C\frac{dv_C}{dt} + v_C = v_{TH}$$



Case 1.2
Current through inductor

Use KVL

$$v_R + v_L = v_{TH}$$

$$v_R = R_{TH} i_L$$

$$v_L = L\frac{di_L}{dt}$$

$$L\frac{di_L}{dt} + R_{TH} i_L = v_{TH}$$

$$\left(\frac{L}{R_{TH}}\right)\frac{di_L}{dt} + i_L = \frac{v_{TH}}{R_{TH}} = i_{SC}$$