VARIABLE-FREQUENCY NETWORK PERFORMANCE

Resonant Circuits

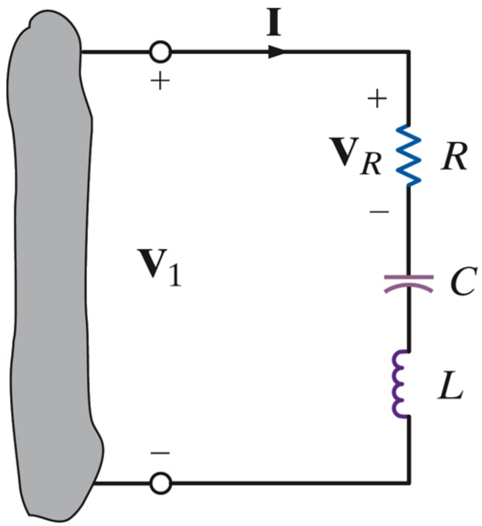
The resonance phenomenon and its characterization

Filter Networks

Networks with frequency selective characteristics: low-pass, high-pass, band-pass

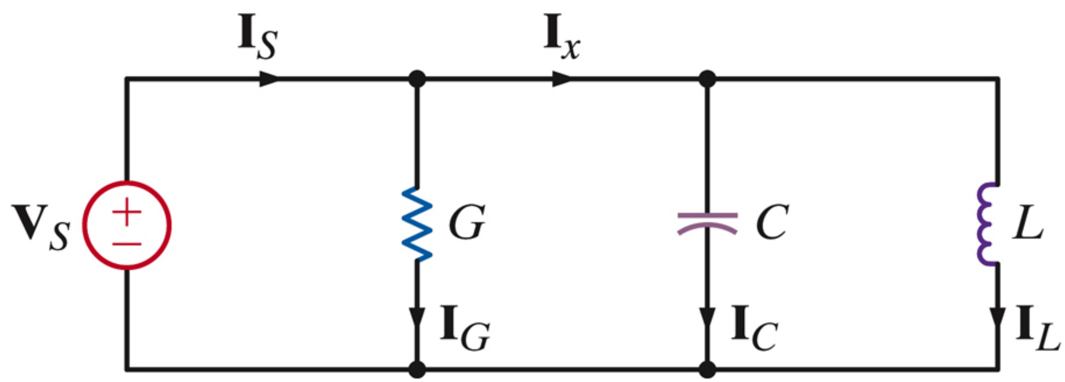
RESONANT CIRCUITS

These are circuits with very special frequency characteristics…
And resonance is a very important physical phenomenon



Series RLC circuit

$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$



Parallel RLC circuit

$$Y(j\omega) = G + j\omega C + \frac{1}{j\omega L}$$

The reactance of each circuit is zero when

$$\omega L = \frac{1}{\omega C} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

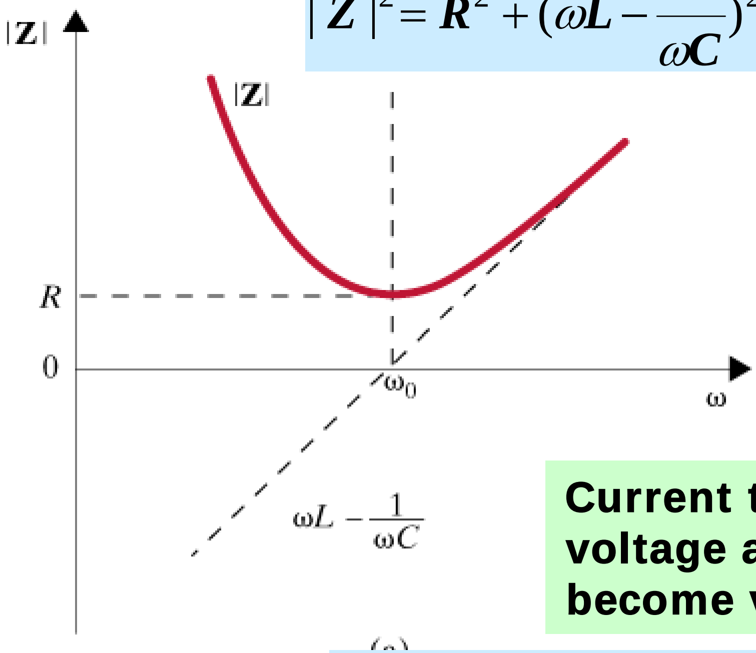
The frequency at which the circuit becomes purely resistive is called the resonance frequency

Properties of resonant circuits

At resonance the impedance/admittance is minimal

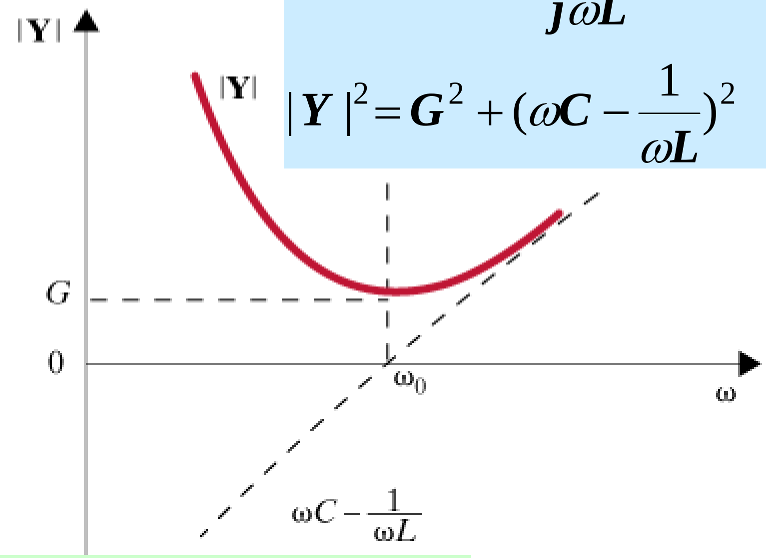
$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$|Z|^2 = R^2 + (\omega L - \frac{1}{\omega C})^2$$



$$Y(j\omega) = G + \frac{1}{j\omega L} + j\omega C$$

$$|Y|^2 = G^2 + (\omega C - \frac{1}{\omega L})^2$$



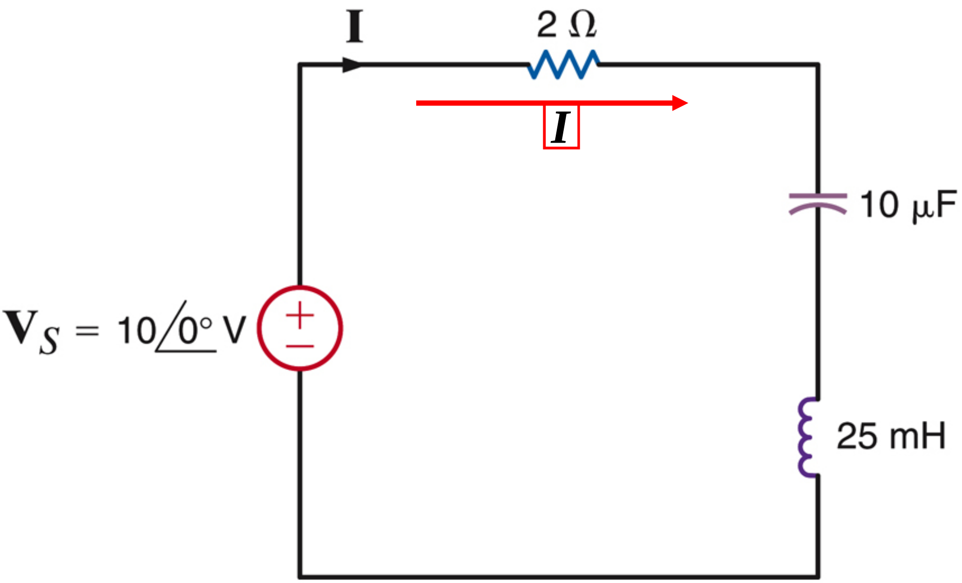
Current through the serial circuit/ voltage across the parallel circuit can become very large (if resistance is small)

Quality Factor: $Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}$

Given the similarities between series and parallel resonant circuits, we will focus on serial circuits

EXAMPLE

Determine the resonant frequency, the voltage across each element at resonance and the value of the quality factor



$$\frac{1}{\omega_0 C} = \omega_0 L = 50\Omega$$

$$V_C = \frac{1}{j\omega_0 C} I = -j50 \times 5 = 250\angle -90^\circ$$

$$Q = \frac{\omega_0 L}{R} = \frac{50}{2} = 25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(25\times10^{-3}H)(10\times10^{-6}F)}} = 2000 rad/\sec$$

At resonance $Z = 2\Omega$

$$I = \frac{V_S}{Z} = \frac{10\angle 0^\circ}{2} = 5A$$

$$\omega_0 L = (2\times10^3)(25\times10^{-3}) = 50\Omega$$

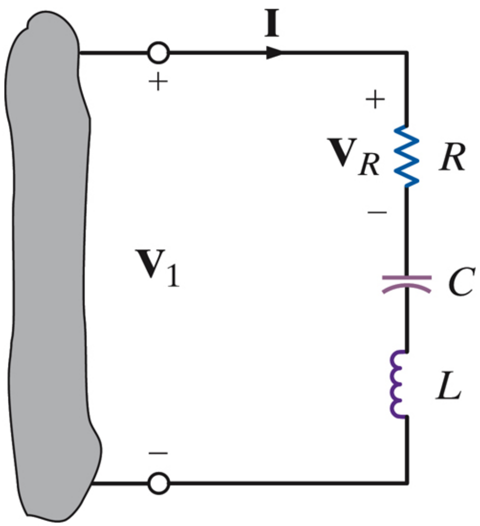
$$V_L = j\omega_0 L I = j50\times5 = 250\angle 90^\circ (V)$$

At resonance

$$|V_L| = \omega_0 L \left|\frac{V_S}{R}\right| = Q|V_S|$$

$$|V_C| = Q|V_S|$$

Resonance for the series circuit



$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$|Z|^2 = R^2 + (\omega L - \frac{1}{\omega C})^2$$

Claim: The voltage gain

$$G_v = \frac{V_R}{V_1} = \frac{1}{1 + jQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})}$$

$$G_v = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{R}{Z(j\omega)}$$

At resonance:

$$\omega_0 L = QR,\ \omega_0 C = \frac{1}{QR}$$

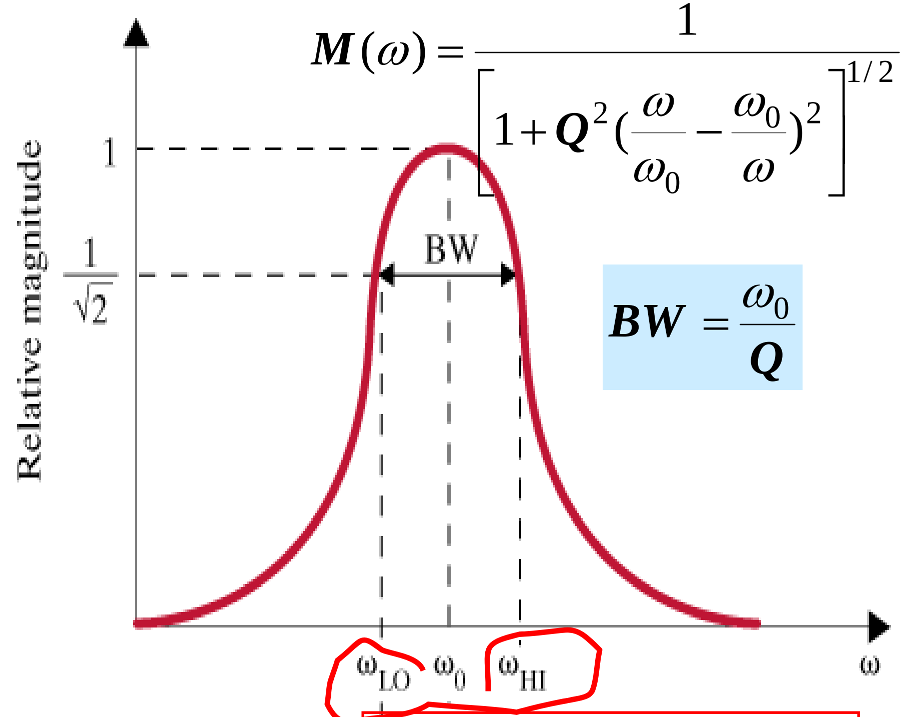
$$Z(j\omega) = R + j\frac{\omega}{\omega_0}QR - j\frac{\omega_0}{\omega}QR$$

$$= R\left[1 + jQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})\right]$$

$$G_v = \frac{R}{Z}$$

$$M(\omega) = |G_v|,\ \phi(\omega) = \angle G_v$$

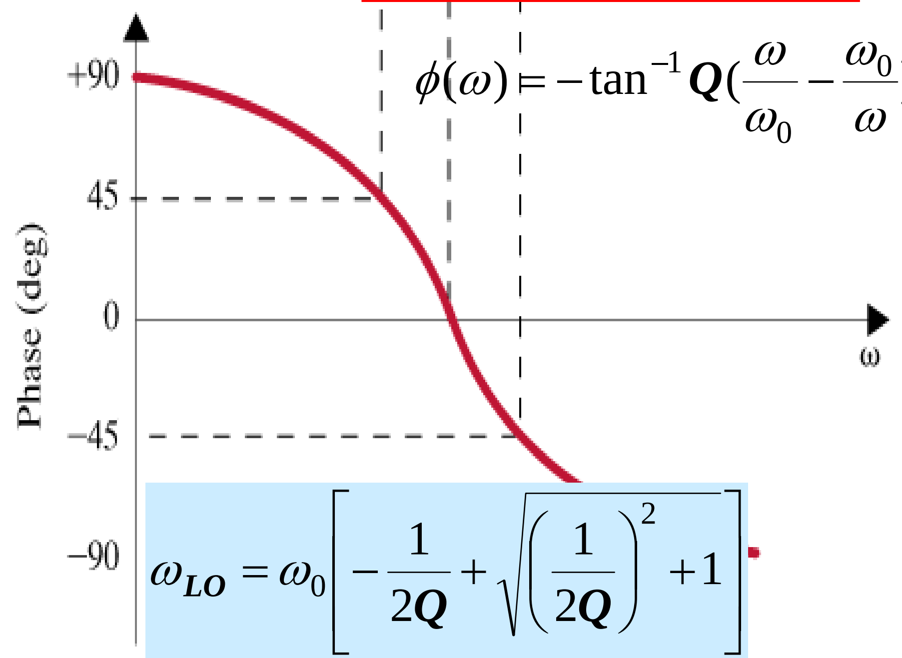
$$M(\omega) = \frac{1}{\left[1 + Q^2(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})^2\right]^{1/2}}$$



$$BW = \frac{\omega_0}{Q}$$

Half power frequencies

$$\phi(\omega) = -\tan^{-1} Q(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})$$



$$\omega_{LO} = \omega_0\left[-\frac{1}{2Q} + \sqrt{\left(\frac{1}{2Q}\right)^2 + 1}\right]$$

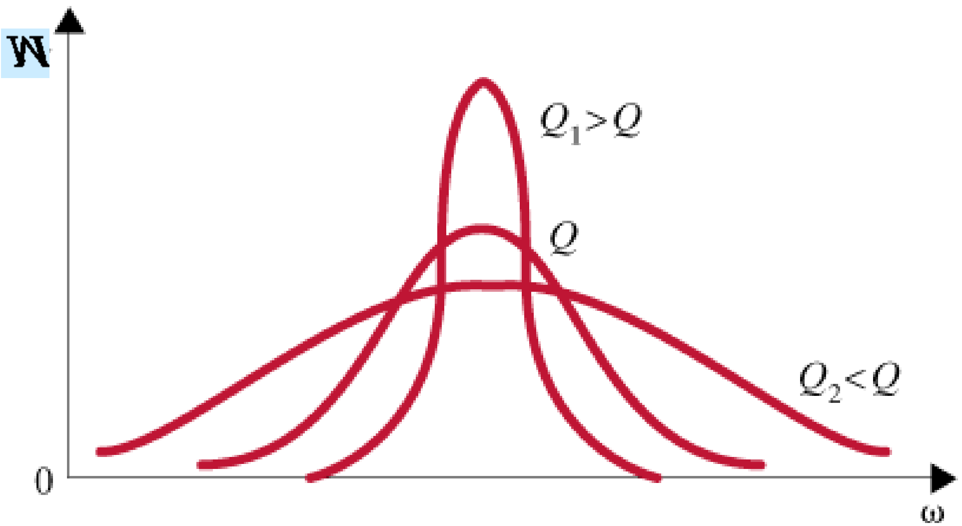
The Q factor

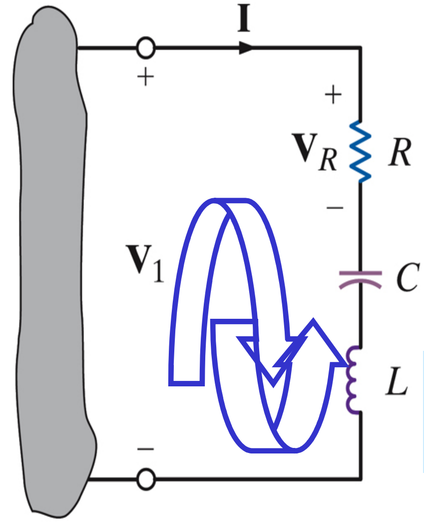
$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

For series circuit : High Q ⇔ Low R

For parallel circuit : High Q ⇔ High R (low G)

High Q ⇔ Small BW





dissipates

Stores as E field

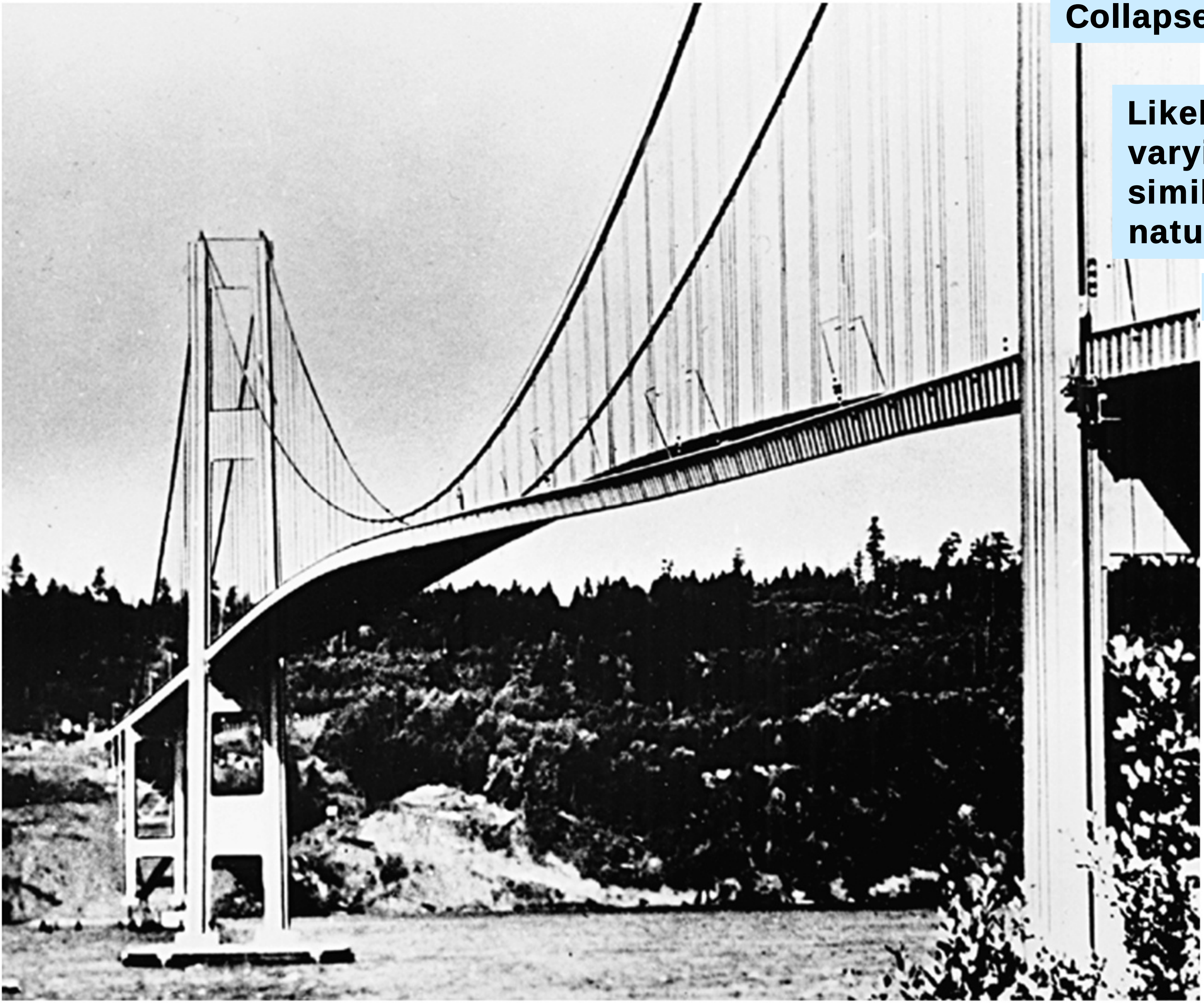
Stores as M field

Capacitor and inductor exchange stored energy. When one is at maximum the other is at zero

EXAMPLE

The Tacoma Narrows Bridge

Opened: July 1, 1940
Collapsed: Nov 7, 1940



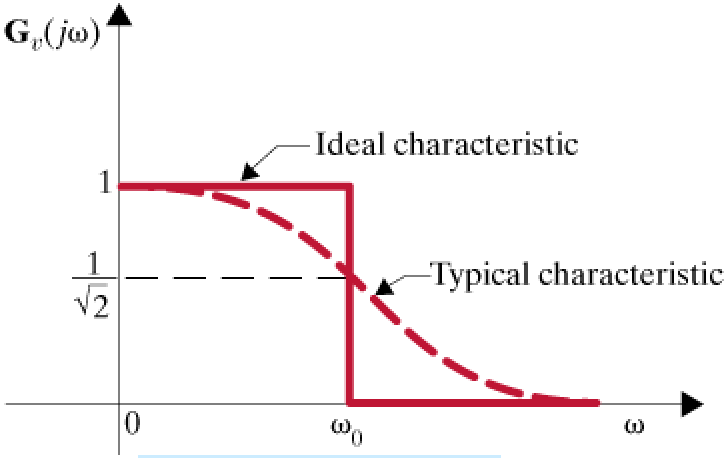
Likely cause: wind varying at frequency similar to bridge natural frequency

$\omega_0 = 2\pi \times 0.2$

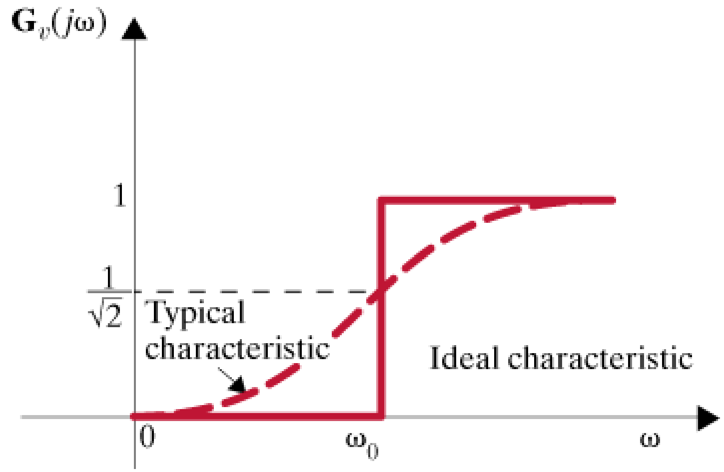
FILTER NETWORKS

Networks designed to have frequency selective behavior

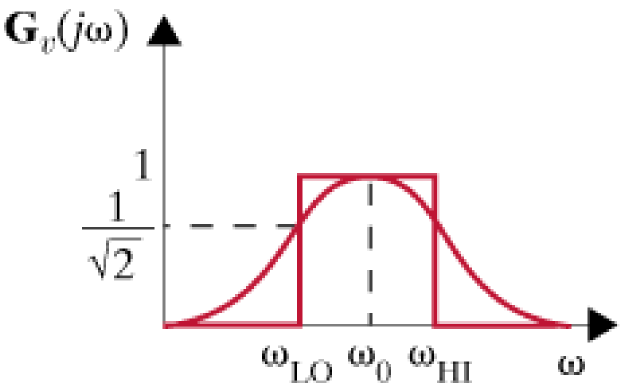
COMMON FILTERS



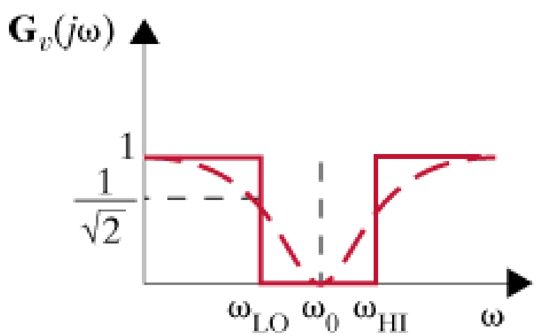
Low-pass filter



High-pass filter

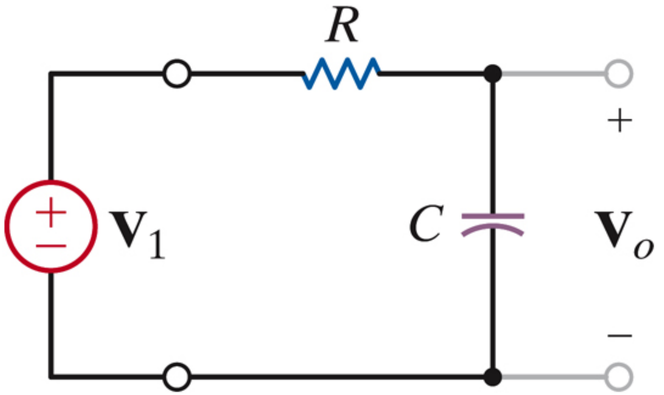


Band-pass filter



Band-reject filter

Simple low-pass filter



$$G_v = \frac{V_0}{V_1} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

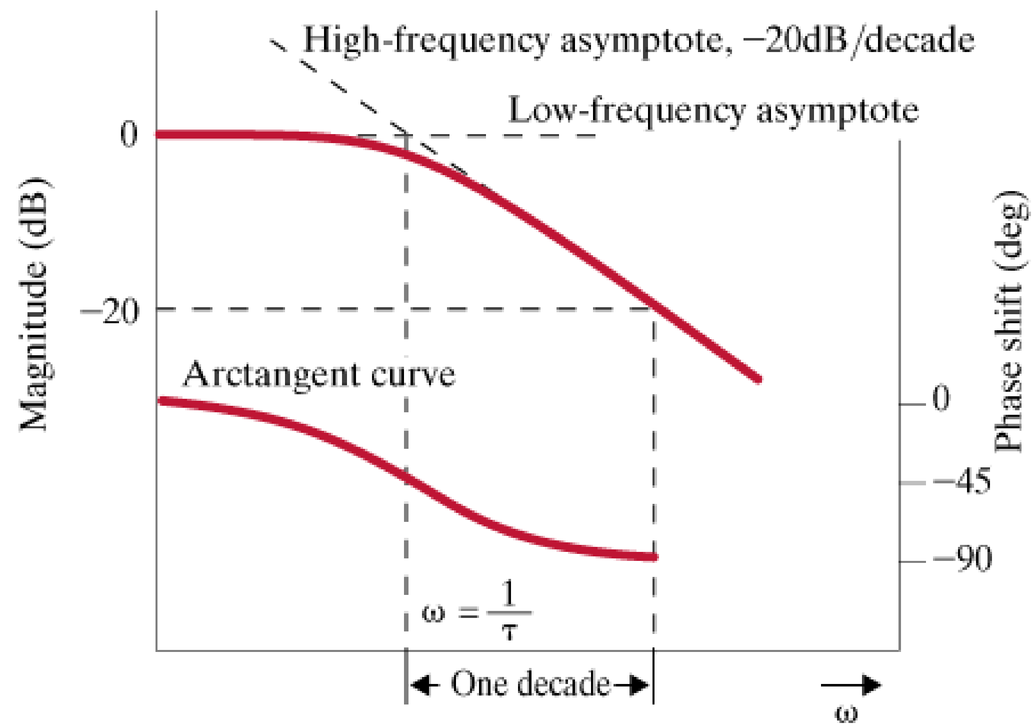
$$G_v = \frac{1}{1 + j\omega\tau}; \quad \tau = RC$$

$$M(\omega) = |G_v| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

$$\angle G_v = \phi(\omega) = -\tan^{-1}\omega\tau$$

$$M_{\max} = 1, \ M\left(\omega = \frac{1}{\tau}\right) = \frac{1}{\sqrt{2}}$$

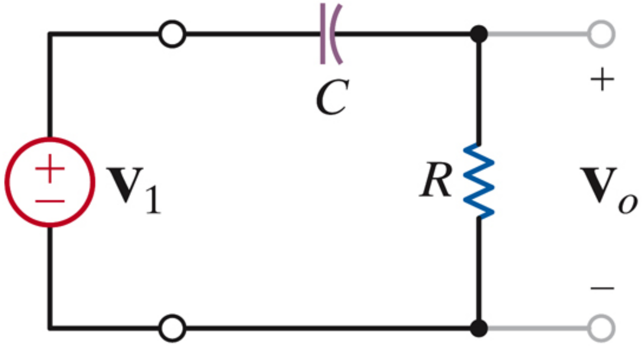
$$\omega = \frac{1}{\tau} = \text{half power frequency}$$



(c)

$$BW = \frac{1}{\tau}$$

Simple high-pass filter



$$G_v = \frac{V_0}{V_1} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega CR}{1 + j\omega CR}$$

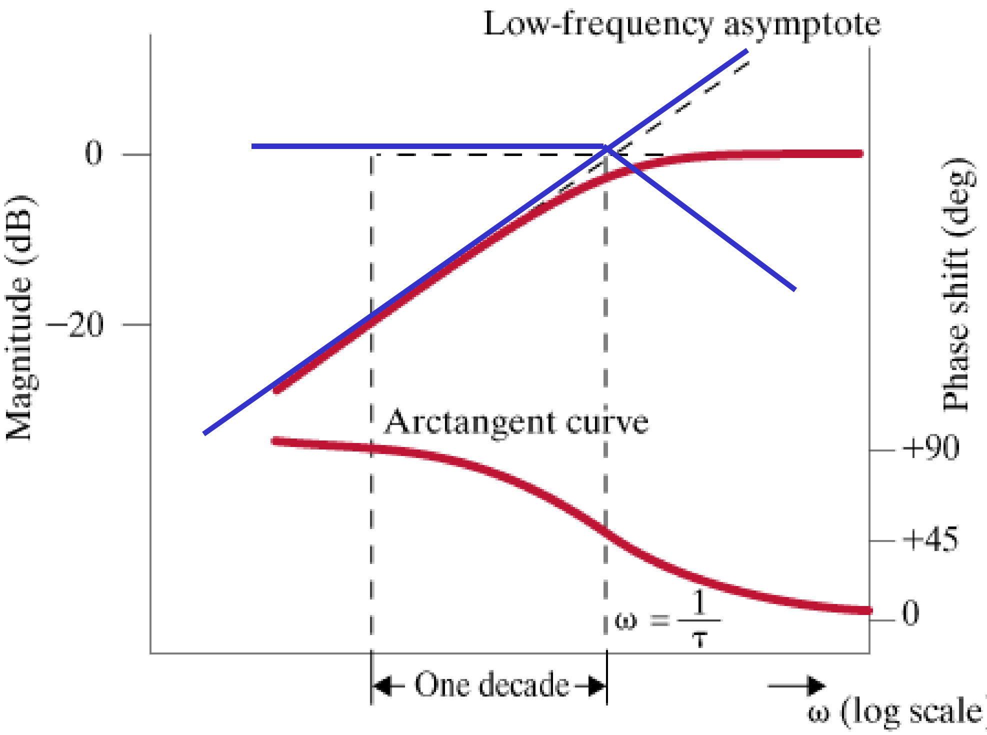
$$G_v = \frac{j\omega\tau}{1 + j\omega\tau}; \ \tau = RC$$

$$M(\omega) = |G_v| = \frac{\omega\tau}{\sqrt{1 + (\omega\tau)^2}}$$

$$\angle G_v = \phi(\omega) = \frac{\pi}{2} - \tan^{-1}\omega\tau$$

$$M_{max} = 1, \ M\left(\omega = \frac{1}{\tau}\right) = \frac{1}{\sqrt{2}}$$

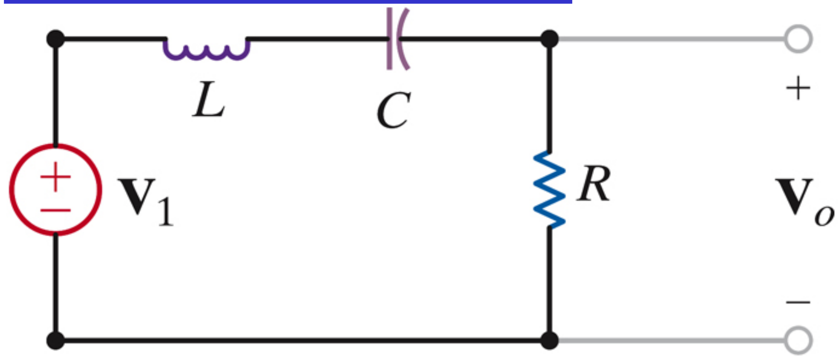
$$\omega = \frac{1}{\tau} = \text{half power frequency}$$



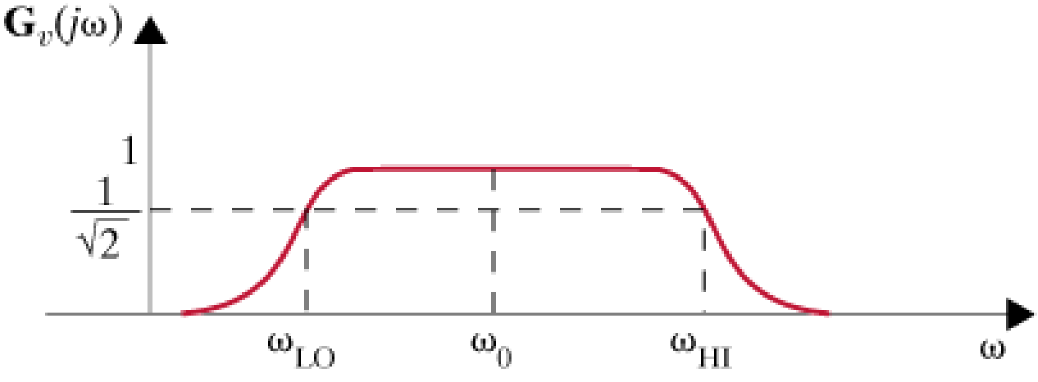
(c)

$$\omega_{LO} = \frac{1}{\tau}$$

Simple band-pass filter



Band-pass



(e)

$$G_v = \frac{V_0}{V_1} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$M(\omega) = \frac{\omega RC}{\sqrt{(\omega RC)^2 + (\omega^2 LC - 1)^2}}$$

$$M\left(\omega = \frac{1}{\sqrt{LC}}\right) = 1 \quad M(\omega = 0) = M(\omega = \infty) = 0$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

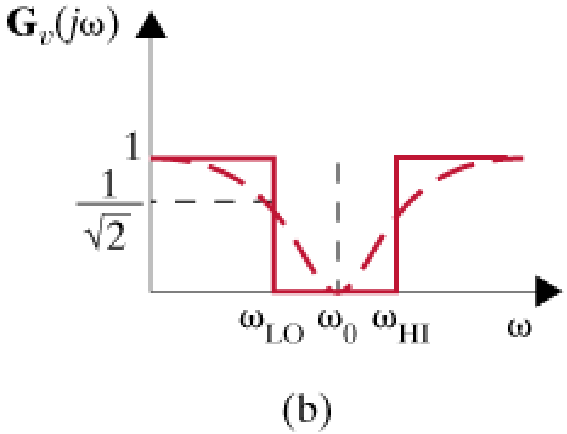
$$M(\omega_{LO}) = \frac{1}{\sqrt{2}} = M(\omega_{HI})$$

$$\omega_{LO} = \frac{-(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2}$$

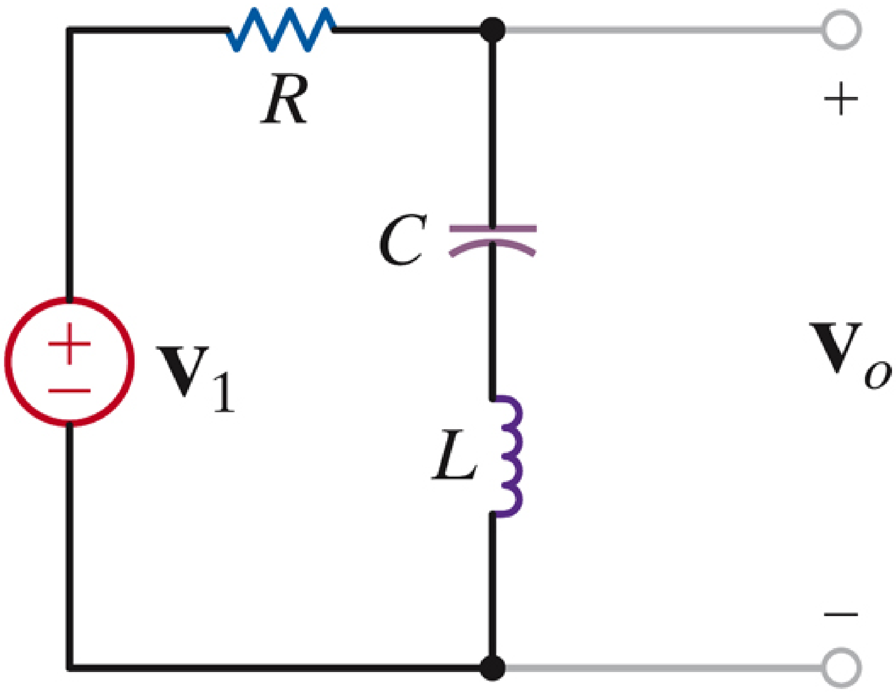
$$\omega_{HI} = \frac{(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2}$$

$$BW = \omega_{HI} - \omega_{LO} = \frac{R}{L}$$

Simple band-reject filter



(b)



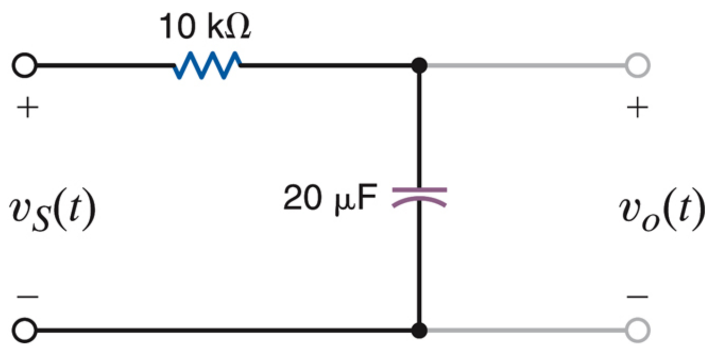
$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow j\left(\omega_0 L - \frac{1}{\omega_0 C}\right) = 0$$

at $\omega = 0$ the capacitor acts as open circuit $\Rightarrow V_0 = V_1$

at $\omega = \infty$ the inductor acts as open circuit $\Rightarrow V_0 = V_1$

ω_{LO}, ω_{HI} are determined as in the band-pass filter

Sketch the magnitude characteristic of the Bode plot for $G_v(j\omega)$



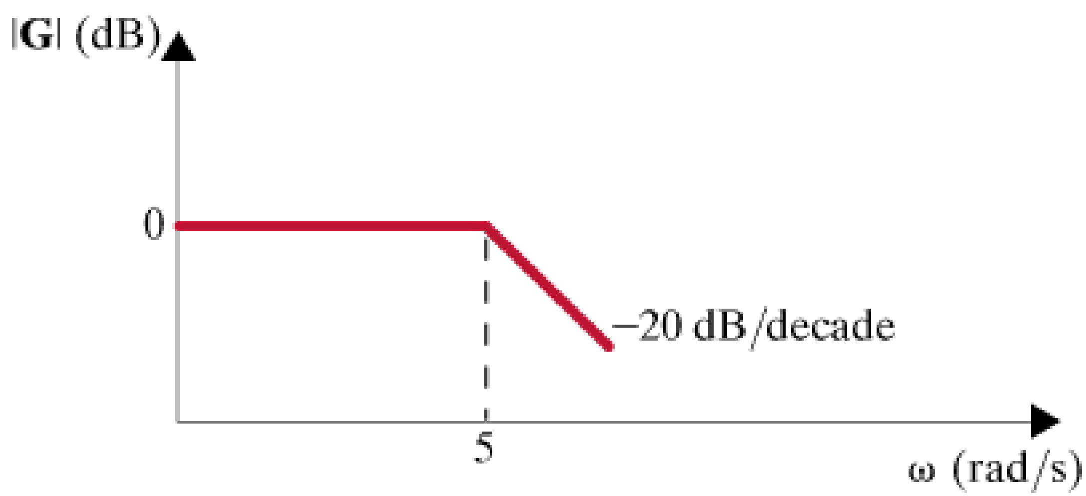
$$G_v(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

$$\tau = RC = (10 \times 10^3 \Omega)(20 \times 10^{-6} F) = 0.2 rad/s$$

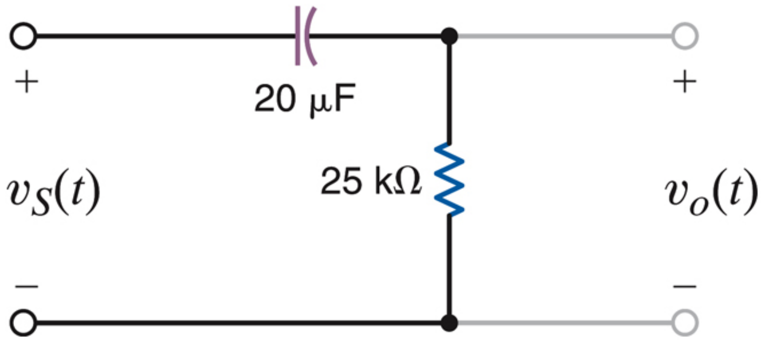
Break/corner frequency : 5rad/s

low frequency asymptote of 0dB/dec

High frequency asymptote of -20dB/dec



Sketch the magnitude characteristic of the Bode plot for $G_v(j\omega)$



$\tau = RC = (25\times10^3\Omega)(20\times10^{-6}F) = 0.5 rad/s$

20dB/dec. Crosses 0dB at $\omega = \frac{1}{\tau} = 2 rad/s$

$$G_v(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC}$$

Break/corner frequency : 2rad/s

low frequency asymptote of 0dB/dec

High frequency asymptote of -20dB/dec