VARIABLE-FREQUENCY NETWORK PERFORMANCE

Variable-Frequency Response Analysis

Network performance as function of frequency.

Transfer function

Sinusoidal Frequency Analysis

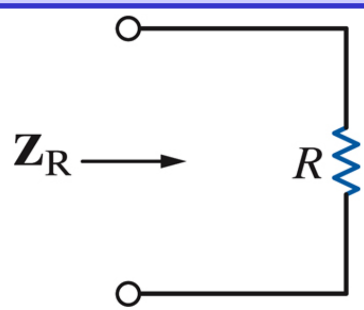
Bode plots to display frequency response data

VARIABLE FREQUENCY-RESPONSE ANALYSIS

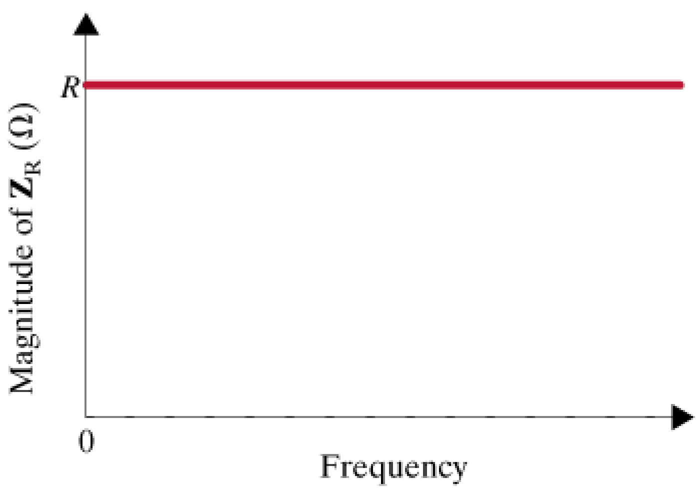
In AC steady state analysis the frequency is assumed constant (e.g., 60Hz). Here we consider the frequency as a variable and examine how the performance varies with the frequency.

Variation in impedance of basic components

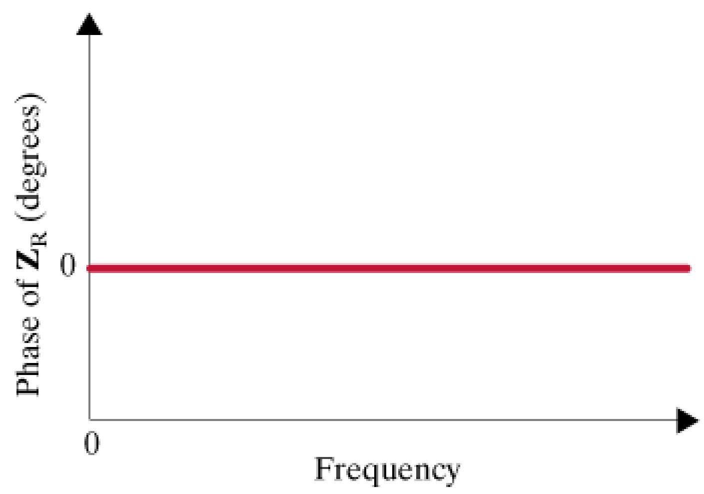
Resistor



$$Z_R = R = R\angle 0^\circ$$

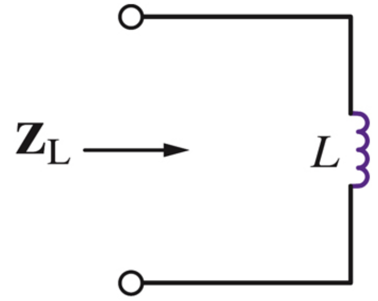


(b)

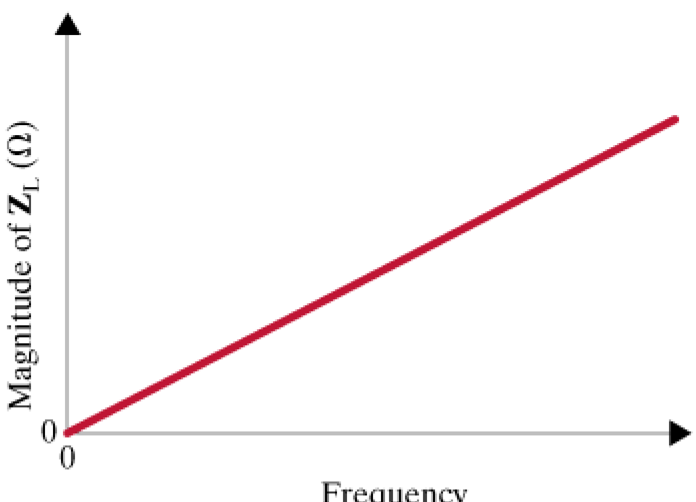


(c)

Inductor



$$Z_L = j\omega L = \omega L \angle 90°$$



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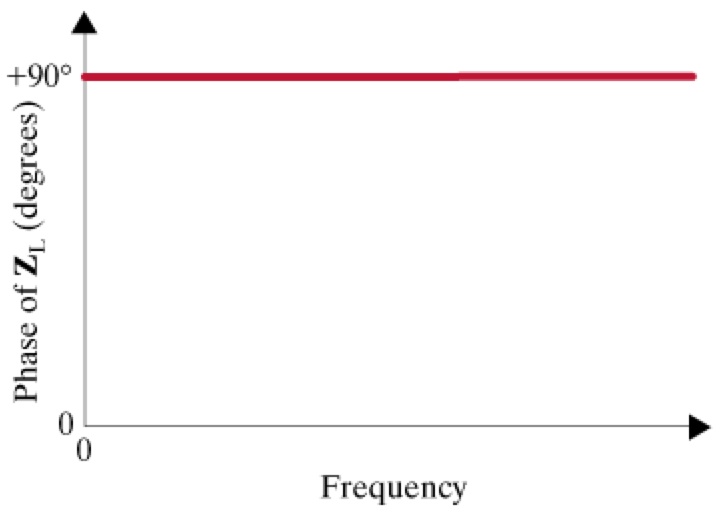
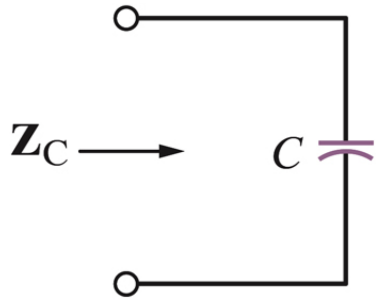
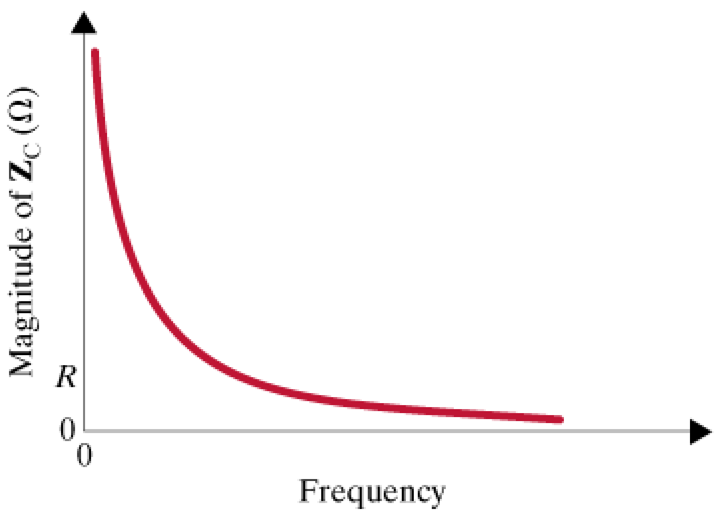


(c)

Capacitor



$$\boldsymbol{Z_c} = \frac{1}{j\omega C} = \frac{1}{\omega C}\angle -90^\circ$$



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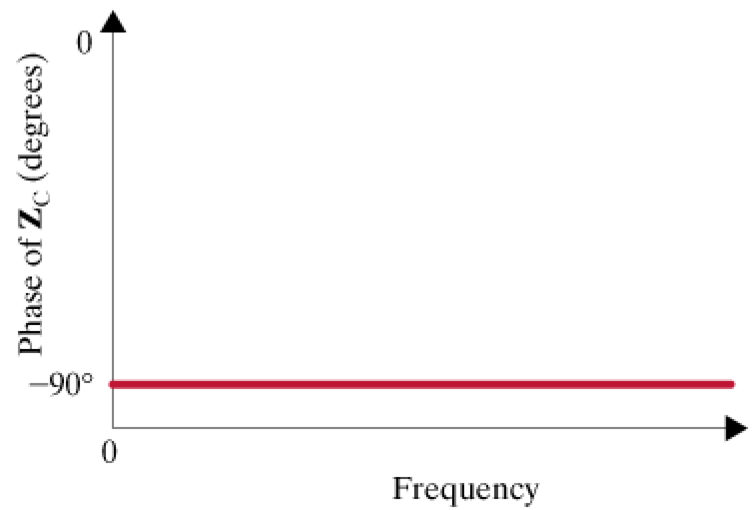
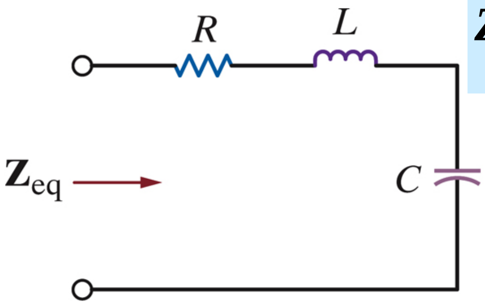


(c)

Frequency dependent behavior of series RLC network



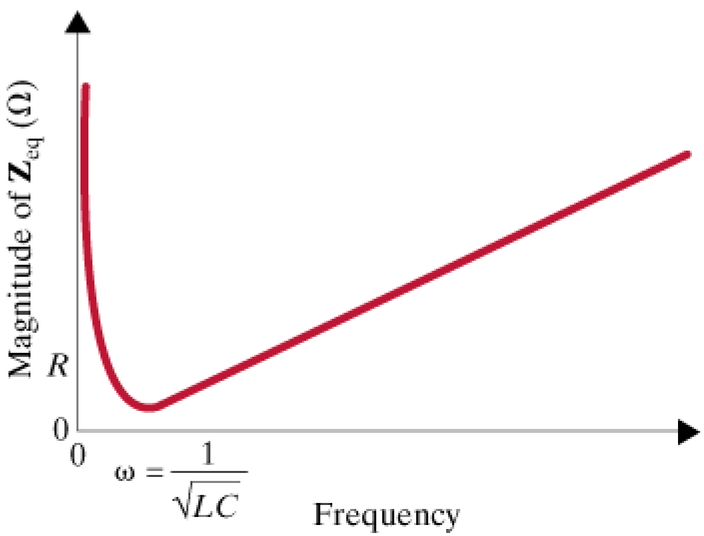
$$Z_{eq} = R + j\omega L + \frac{1}{j\omega C} = \frac{(j\omega)^2 LC + j\omega RC + 1}{j\omega C} \times \frac{-j}{-j} = \frac{\omega RC + j(\omega^2 LC - 1)}{\omega C}$$

"Simplification in notation" $j\omega \approx s$

$$Z_{eq}(s) = \frac{s^2 LC + sRC + 1}{sC}$$

$$|Z_{eq}| = \frac{\sqrt{(\omega RC)^2 + (\omega^2 LC - 1)^2}}{\omega C}$$

$$\angle Z_{eq} = \tan^{-1}\left(\frac{\omega^2 LC - 1}{\omega RC}\right)$$



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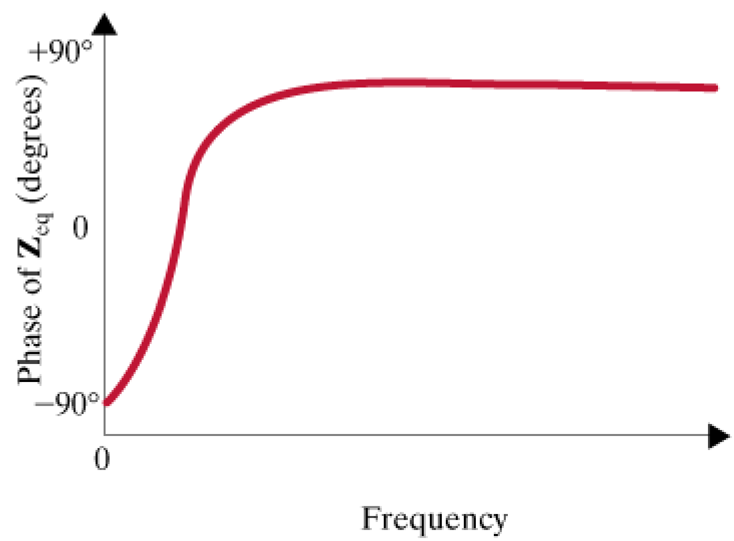


(c)

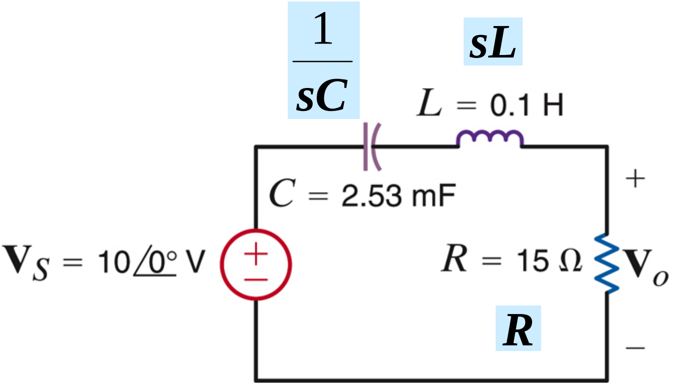
Simplified notation for basic components

$$Z_R(s) = R,\ \ Z_L(s) = sL,\ \ Z_C = \frac{1}{sC}$$

For all cases seen, and all cases to be studied, the impedance is of the form

$$Z(s) = \frac{a_m s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}$$

Moreover, if the circuit elements (L,R,C, dependent sources) are real then the expression for any voltage or current will also be a rational function in *s*



$$V_o(s) = \frac{R}{R + sL + 1/sC}V_S = \frac{sRC}{s^2LC + sRC + 1}V_S$$

$$s \approx j\omega$$

$$V_o = \frac{j\omega RC}{(j\omega)^2 LC + j\omega RC + 1}V_S$$

$$V_o = \frac{j\omega(15\times 2.53\times 10^{-3})}{(j\omega)^2(0.1\times 2.53\times 10^{-3}) + j\omega(15\times 2.53\times 10^{-3}) + 1}10\angle 0°$$

NETWORK FUNCTIONS

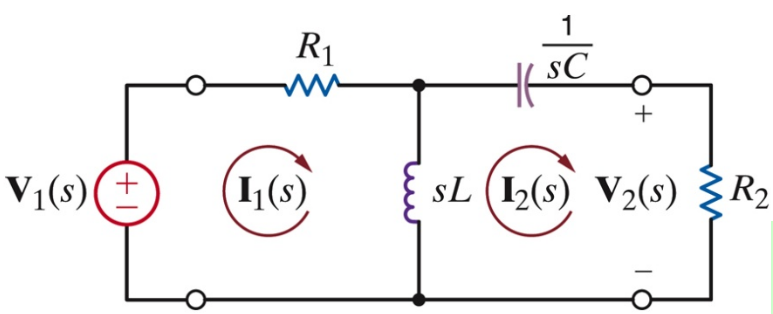
Some nomenclature

When voltages and currents are defined at different terminal pairs we define the ratios as Transfer Functions

INPUT	OUTPUT	TRANSFER FUNCTION	SYMBOL
Voltage	Voltage	Voltage Gain	**Gv(s)**
Current	Voltage	Transimpedance	**Z(s)**
Current	Current	Current Gain	**Gi(s)**
Voltage	Current	Transadmittance	**Y(s)**

If voltage and current are defined at the same terminals we define Driving Point Impedance/Admittance

EXAMPLE

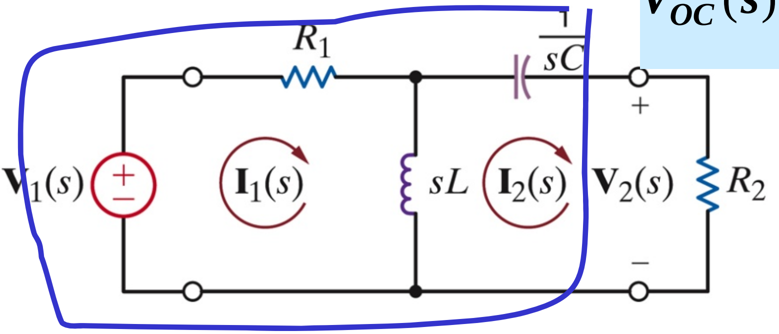


To compute the transfer functions one must solve the circuit. Any valid technique is acceptable

$$Y_T(s) = \frac{I_2(s)}{V_1(s)} \begin{cases} \text{Transadmittance} \\ \text{Transfer admittance} \end{cases}$$

$$G_v(s) = \frac{V_2(s)}{V_1(s)} \quad \text{Voltage gain}$$

EXAMPLE



$$V_{OC}(s) = \frac{sL}{sL + R_1} V_1(s)$$

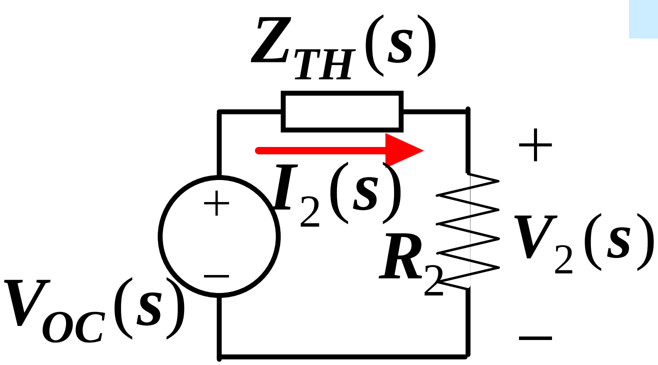
We will use Thevenin's theorem

$$Z_{TH}(s) = \frac{1}{sC} + R_1 \,\|\, sL = \frac{1}{sC} + \frac{sLR_1}{sL + R_1}$$

$$Z_{TH}(s) = \frac{s^2 LCR_1 + sL + R_1}{sC(sL + R_1)}$$

$$Y_T(s) = \frac{I_2(s)}{V_1(s)} \begin{cases} \text{Transadmittance} \\ \text{Transfer admittance} \end{cases}$$

$$G_v(s) = \frac{V_2(s)}{V_1(s)} \quad \text{Voltage gain}$$



$$I_2(s) = \frac{V_{OC}(s)}{R_2 + Z_{TH}(s)} = \frac{\frac{sL}{sL + R_1} V_1(s)}{R_2 + \frac{s^2 LCR_1 + sL + R_1}{sC(sL + R_1)}} \times \frac{sC(sL + R_1)}{sC(sL + R_1)}$$

$$Y_T(s) = \frac{s^2 LC}{s^2 (R_1 + R_2) LC + s(L + R_1 R_2 C) + R_1}$$

$$G_v(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_2 I_2(s)}{V_1(s)} = R_2 Y_T(s)$$

POLES AND ZEROS

(More nomenclature)

$$H(s) = \frac{a_m s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}$$

Arbitrary network function

Using the roots, every (monic) polynomial can be expressed as a product of first order terms

$$H(s) = K_0 \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

$z_1, z_2, \ldots, z_m =$ zeros of the network function

$p_1, p_2, \ldots, p_n =$ poles of the network function

The network function is uniquely determined by its poles and zeros and its value at some other value of s (to compute the gain)

EXAMPLE

zeros : $z_1 = -1,$

poles : $p_1 = -2 + j2, p_2 = -2 - j2$

$H(0) = 1$

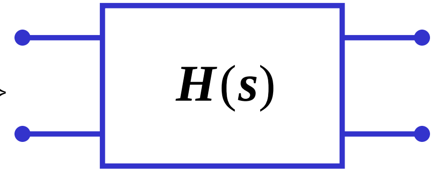
$$H(s) = K_0 \frac{(s+1)}{(s+2-j2)(s+2+j2)} = K_0 \frac{s+1}{s^2 + 4s + 8}$$

$$H(0) = K_0 \frac{1}{8} = 1 \Rightarrow$$

$$H(s) = 8\frac{s+1}{s^2 + 4s + 8}$$

SINUSOIDAL FREQUENCY ANALYSIS

$$\left.\begin{array}{r} A_0 e^{j(\omega t+\theta)} \\ B_0 \cos(\omega t+\theta) \end{array}\right\}$$



$$\left\{\begin{array}{l} A_0 H(j\omega) e^{j(\omega t+\theta)} \\ B_0 \,|\, H(j\omega) \,|\, \cos(\omega t+\theta+\angle H(j\omega)) \end{array}\right.$$

Circuit represented by network function

To study the behavior of a network as a function of the frequency we analyze the network function $H(j\omega)$ as a function of ω.

Notation

$$M(\omega) = |H(j\omega)|$$

$$\phi(\omega) = \angle H(j\omega)$$

$$H(j\omega) = M(\omega) e^{j\phi(\omega)}$$

Plots of $M(\omega), \phi(\omega)$, as function of ω are generally called magnitude and phase characteristics.

BODE PLOTS $\left\{\begin{array}{c} 20\log_{10}(M(\omega)) \\ \phi(\omega) \end{array}\right.$ vs $\log_{10}(\omega)$

HISTORY OF THE DECIBEL

Originated as a measure of relative (radio) power

$$\boldsymbol{P}_2\big|_{dB}\ (\text{over } P_1) = 10\log\frac{\boldsymbol{P}_2}{\boldsymbol{P}_1}$$

$$\boldsymbol{P} = \boldsymbol{I}^2\boldsymbol{R} = \frac{\boldsymbol{V}^2}{\boldsymbol{R}} \Rightarrow \boldsymbol{P}_2\big|_{dB}\ (\text{over } P_1) = 10\log\frac{\boldsymbol{V}_2^2}{\boldsymbol{V}_1^2} = 10\log\frac{\boldsymbol{I}_2^2}{\boldsymbol{I}_1^2}$$

By extension

$$\boldsymbol{V}\big|_{dB} = 20\log_{10}|\boldsymbol{V}|$$

$$\boldsymbol{I}\big|_{dB} = 20\log_{10}|\boldsymbol{I}|$$

$$\boldsymbol{G}\big|_{dB} = 20\log_{10}|\boldsymbol{G}|$$

Using log scales the frequency characteristics of network functions have simple asymptotic behavior.
The asymptotes can be used as reasonable and efficient approximations

General form of a network function showing basic terms

Frequency independent

Poles/zeros at the origin

$$H(j\omega) = \frac{K_0 (j\omega)^{\pm N} (1 + j\omega\tau_1)[1 + 2\varsigma_3(j\omega\tau_3) + (j\omega\tau_3)^2]\ldots}{(1 + j\omega\tau_a)[1 + 2\varsigma_b(j\omega\tau_b) + (j\omega\tau_b)^2]\ldots}$$

First order terms

Quadratic terms for complex conjugate poles/zeros

$$\log(AB) = \log A + \log B$$

$$\log\left(\frac{N}{D}\right) = \log N - \log D$$

$$|H(j\omega)|_{dB} = 20\log_{10}|H(j\omega)| = 20\log_{10} K_0 \pm N\,20\log_{10}|j\omega|$$
$$+ 20\log_{10}|1 + j\omega\tau_1| + 20\log_{10}|1 + 2\varsigma_3(j\omega\tau_3) + (j\omega\tau_3)^2| + \ldots$$
$$- 20\log_{10}|1 + j\omega\tau_a| - 20\log_{10}|1 + 2\varsigma_b(j\omega\tau_b) + (j\omega\tau_b)^2| - ..$$

$$\angle z_1 z_2 = \angle z_1 + \angle z_2$$

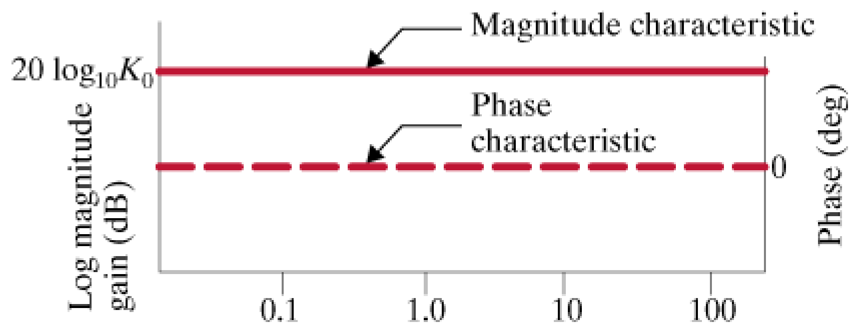
$$\angle \frac{z_1}{z_2} = \angle z_1 - \angle z_2$$

$$\angle H(j\omega) = 0 \pm N 90^\circ$$
$$+ \tan^{-1}\omega\tau_1 + \tan^{-1}\frac{2\varsigma_3\omega\tau_3}{1 - (\omega\tau_3)^2} + \ldots$$
$$- \tan^{-1}\omega\tau_a - \tan^{-1}\frac{2\varsigma_b\omega\tau_b}{1 - (\omega\tau_b)^2} - \ldots$$

Display each basic term separately and add the results to obtain final answer

Let's examine each basic term

Constant Term



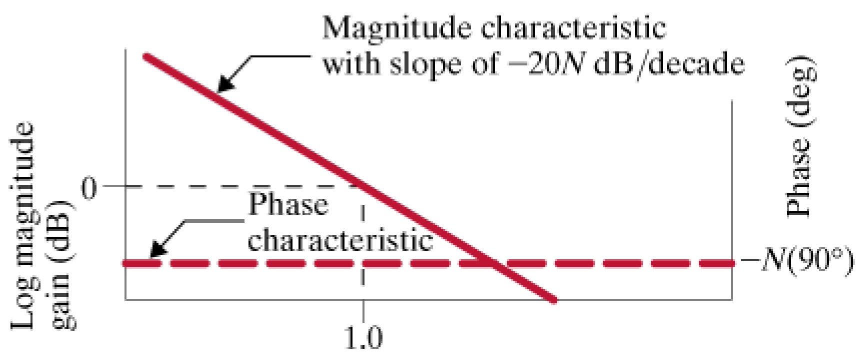
(a)

the x - axis is $\log_{10}\omega$

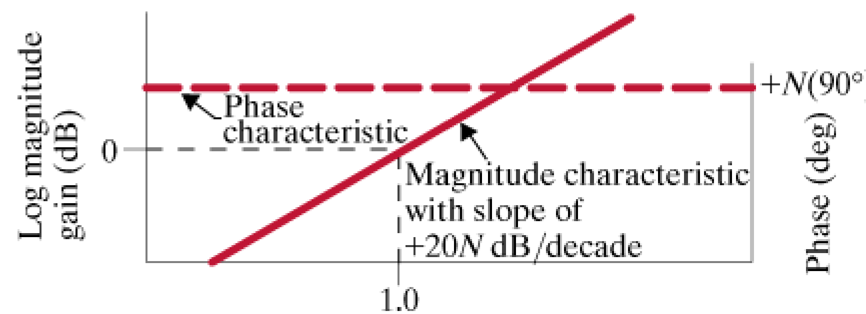
this is a straight line

Poles/Zeros at the origin

$$(j\omega)^{\pm N} \rightarrow \begin{cases} |(j\omega)^{\pm N}|_{dB} = \pm N \times 20\log_{10}(\omega) \\ \angle(j\omega)^{\pm N} = \pm N 90^\circ \end{cases}$$



(b)



(c)

Simple pole or zero $1+j\omega\tau$ $\begin{cases} |1+j\omega\tau|_{dB} = 20\log_{10}\sqrt{1+(\omega\tau)^2} \\ \angle(1+j\omega\tau) = \tan^{-1}\omega\tau \end{cases}$

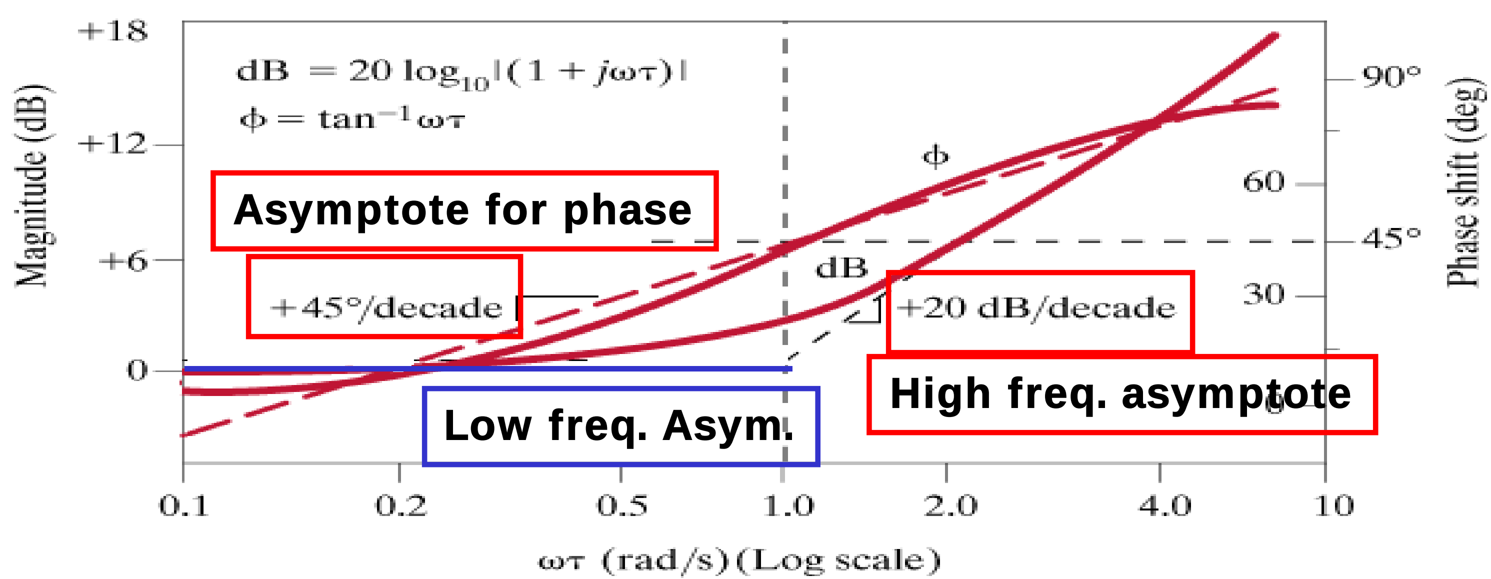
$\omega\tau << 1 \Rightarrow$ $|1+j\omega\tau|_{dB} \approx 0$ low frequency asymptote $\angle(1+j\omega\tau) \approx 0°$

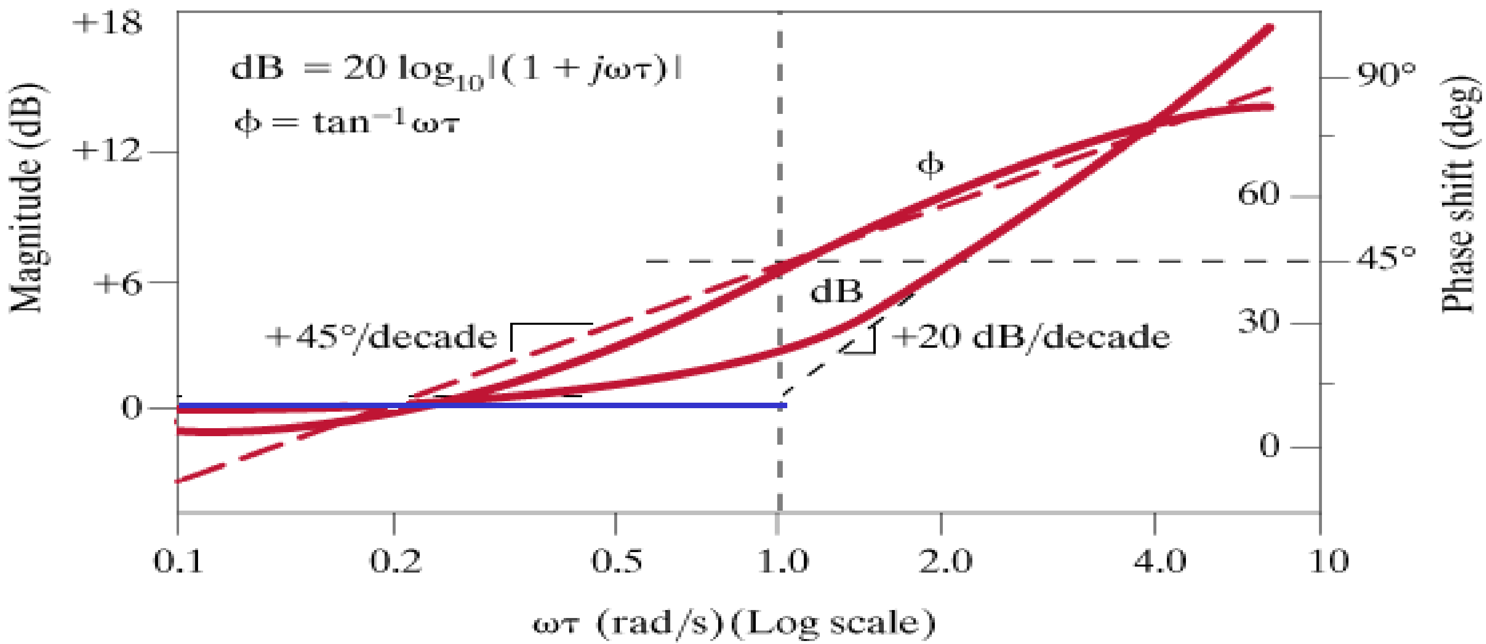
$\omega\tau >> 1 \Rightarrow$ $|1+j\omega\tau|_{dB} \approx 20\log_{10}\omega\tau$ high frequency asymptote (20dB/dec) $\angle(1+j\omega\tau) \approx 90°$

The two asymptotes meet when $\omega\tau = 1$(corner/break frequency)

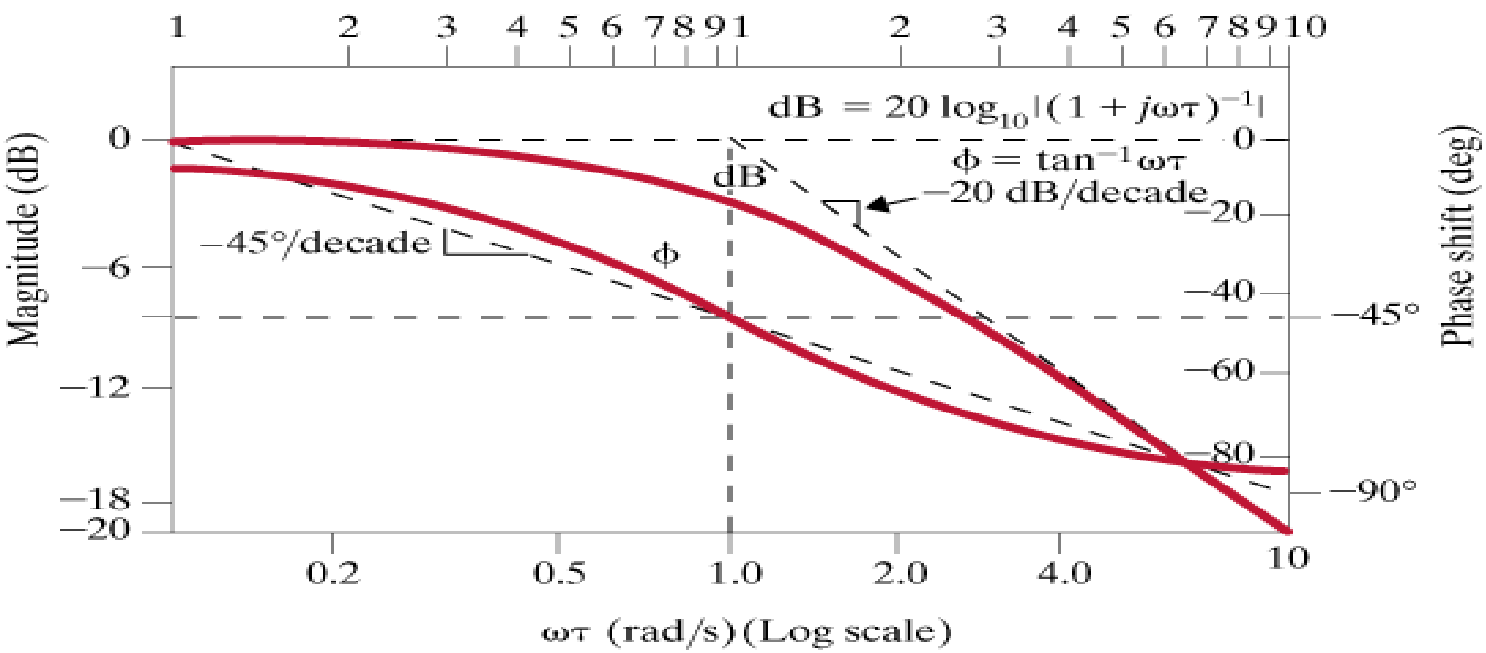
Behavior in the neighborhood of the corner

	Frequency	Asymptote	Curve	distance to asymptote	Argument
corner	$\omega\tau = 1$	0dB	3dB	3	45
octave above	$\omega\tau = 2$	6dB	7db	1	63.4
octave below	$\omega\tau = 0.5$	0dB	1dB	1	26.6





Simple zero



(a)

Simple pole

LEARNING EXAMPLE

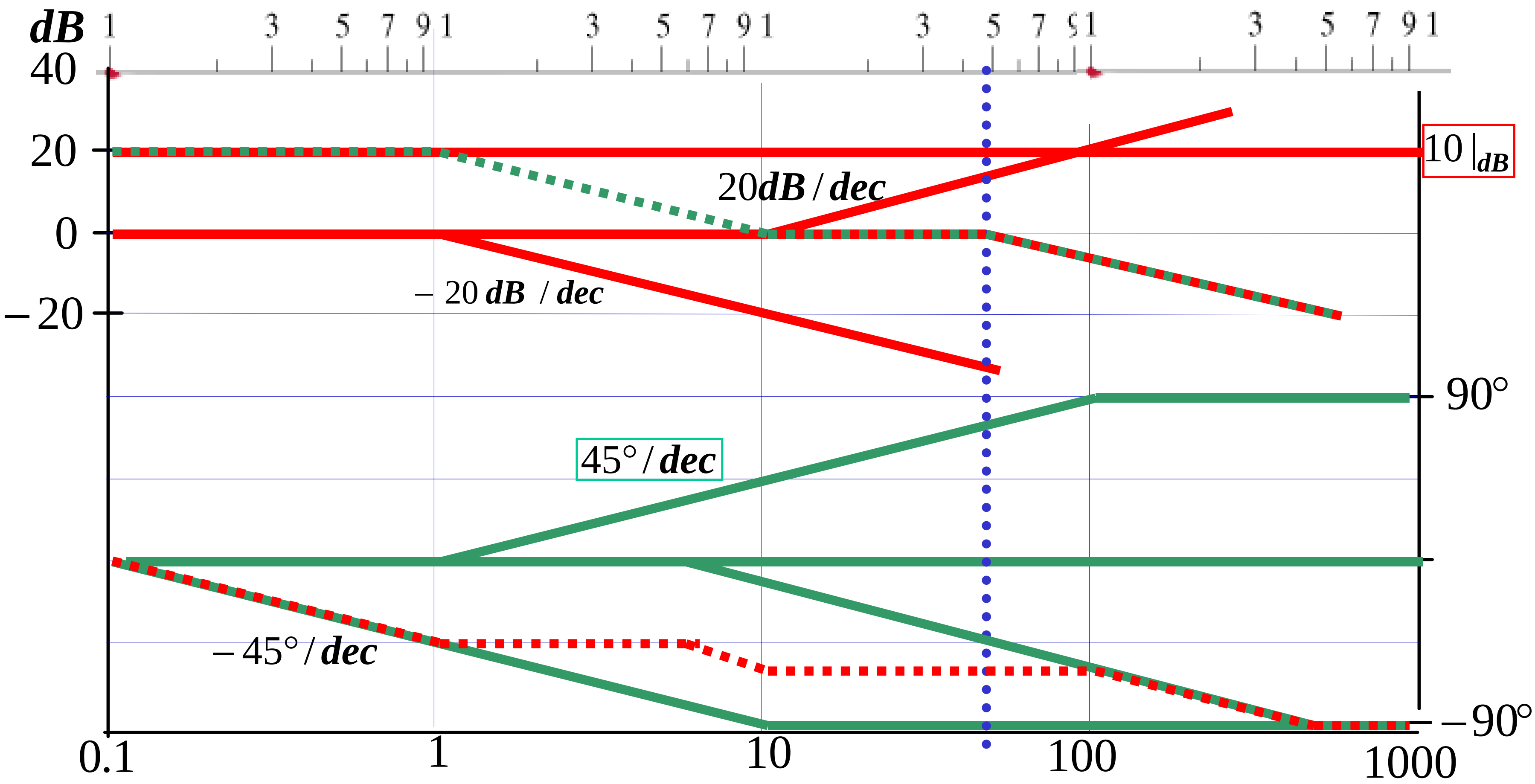
Generate magnitude and phase plots

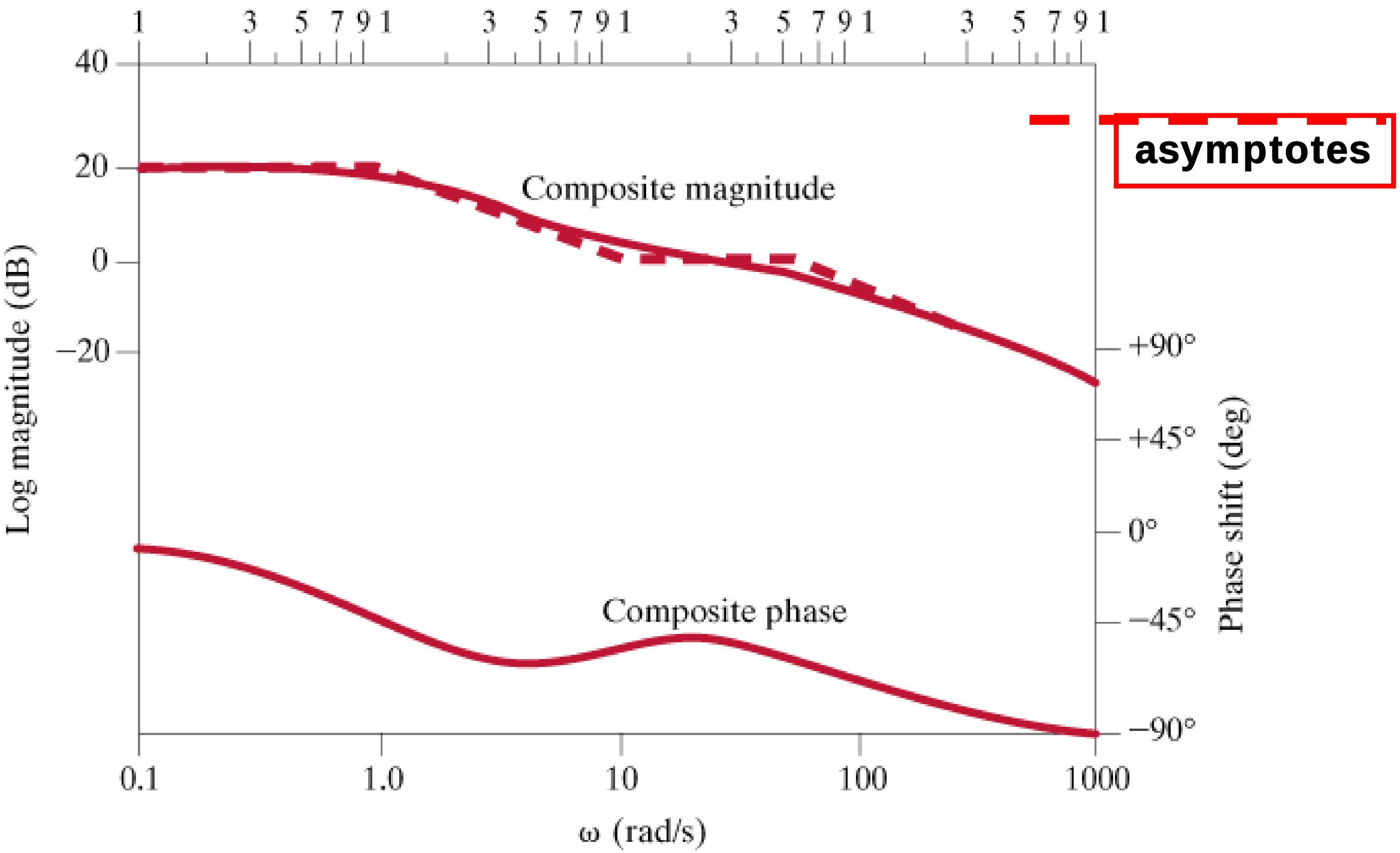
$$G_v(j\omega) = \frac{10(0.1j\omega+1)}{(j\omega+1)(0.02j\omega+1)}$$

Breaks/corners: 1,10,50

Draw asymptotes for each term

Draw composites



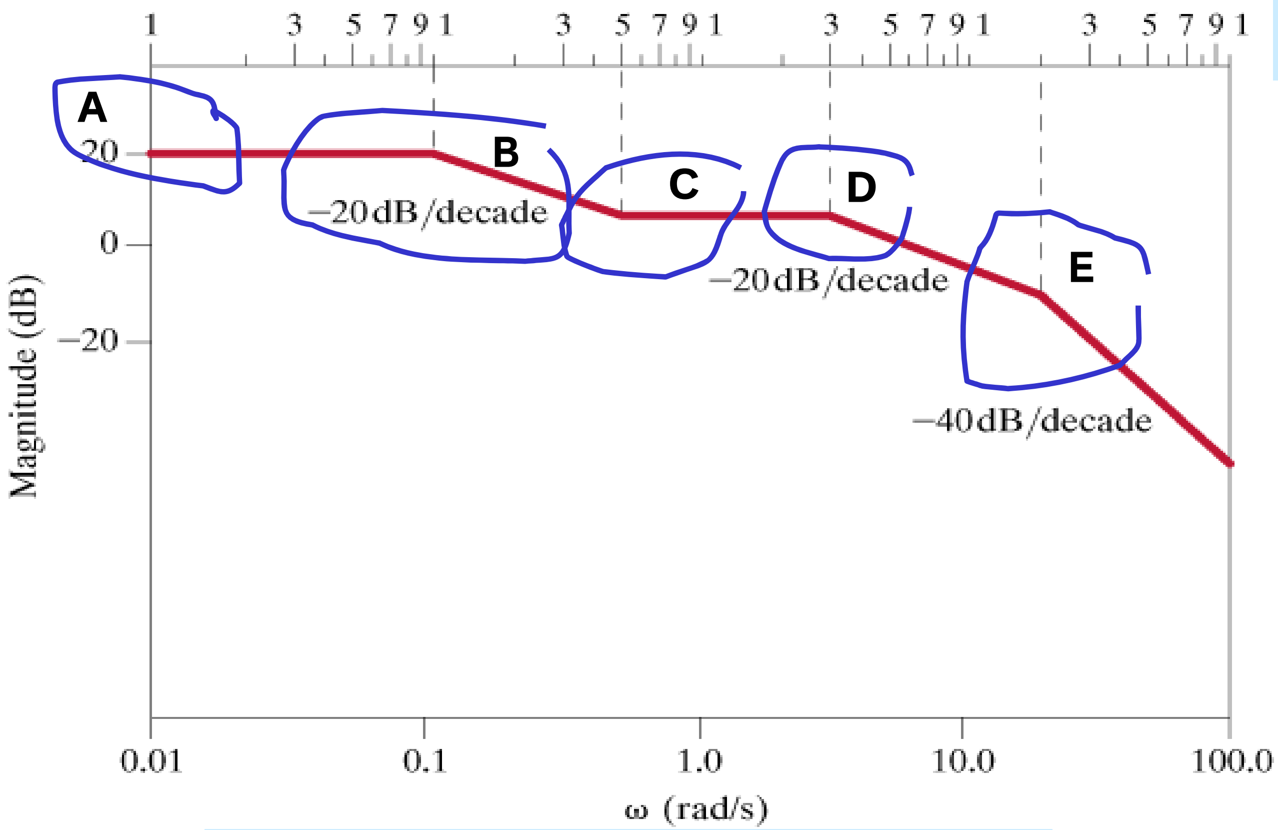


asymptotes
Composite magnitude
Composite phase
Log magnitude (dB)
Phase shift (deg)
40
20
0
−20
+90°
+45°
0°
−45°
−90°
0.1
1.0
10
100
1000
ω (rad/s)

(b)

DETERMINING THE TRANSFER FUNCTION FROM THE BODE PLOT

This is the inverse problem of determining frequency characteristics. We will use only the composite asymptotes plot of the magnitude to postulate a transfer function. The slopes will provide information on the order



A. different from 0dB. There is a constant Ko

$$K_0\,|_{dB} = 20 \Rightarrow K_0 = 10^{\frac{K_0|_{dB}}{20}}$$

B. Simple pole at 0.1

$$(j\omega/0.1+1)^{-1}$$

C. Simple zero at 0.5

$$(j\omega/0.5+1)$$

D. Simple pole at 3

$$(j\omega/3+1)^{-1}$$

E. Simple pole at 20

$$(j\omega/20+1)^{-1}$$

$$G(j\omega) = \frac{10(j\omega/0.5+1)}{(j\omega/0.1+1)(j\omega/3+1)(j\omega/20+1)}$$

If the slope is -40dB we assume double real pole. Unless we are given more data