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LARGE CHANGE SENSITIVITY ANALYSIS IN LINEAR SYSTEMS USING GENERALIZED HOUSEHOLDER FORMULAE

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SUMMARY

This paper investigates multiparameter large change sensitivity problems in linear systems by a set of generalized Householder formulae. The newly developed rectangular formulae can accommodate large, small and zero parameter changes directly by avoiding a critical matrix inversion as compared to the traditional square formulae. Possible determination of a minimum order reduced system, whose solution procedure constitutes the major work in large change evaluation is discussed. Applications to linear systems are considered for the original and adjoint systems w.r.t. single as well as multiple input-output cases. This approach makes it possible to use large change analysis algorithms even if many parameters are changed.

INTRODUCTION

In computer-aided circuit design, it is often required to calculate network responses after the parameters in a certain set are changed. This problem, referred to as large change sensitivity analysis, has been studied by many people. Fidler[1] and Singhal *et al.*[2] considered single and multiple parameter changes, respectively, and developed methods to calculate the response function as a multilinear form in variable parameters. Another method is to formulate a reduced system, whose solutions are then used to update the responses. This method has been treated from different angles, e.g. the current source substitution approach of Leung and Spence,[3] the adjoint network approach of Temes and Cho,[4] the Householder formula approach,[3,5] the scattering matrix approach of Haley[6,7] and the matrix partitioning approach of Vlach and Singhal.[8] Hajj[5] has derived and summarized a set of algorithms where finite, infinite and zero parameter changes are all permitted and sparsity is exploited. A recent overview of this area has been given by Haley and Current[7] who presented general approaches encompassing most of the previous methods.

As already noticed,[3,5] large change analysis algorithms will lose efficiency when too many parameters are changed. This is mainly because the algorithms involve the solution of a reduced system of order n_ϕ, the number of variables. However, cases exist where this system is larger than needed. Also, in a Monte Carlo analysis or in an optimization procedure, it is possible that some variables change slightly while others change substantially. In this case, the small parameter changes may cause ill-conditioning in a non-iterative method[3-5] and the large changes may affect the convergence rate in an iterative method.[5]

In this paper, we present a set of generalized Householder formulae which is capable of handling complicated cases encountered in practice. The problem of determining a minimum reduced system is investigated. Different aspects of the basic set of formulae are discussed in terms of duality property and operation count. Applications to general linear systems are considered for original and adjoint responses with single and multiple input and output situations. Also, as a special case, a series of first-order sensitivity expressions are obtained without reference to Tellegen's theorem. Numerical examples are given for a general system of linear equations and for an arbitrary 10 node electrical circuit.

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A SET OF GENERALIZED HOUSEHOLDER FORMULAE

Let the linear system be characterized by an $n \times n$ matrix $\mathbf{A}$. Suppose the parameters $\boldsymbol{\phi}$ of the system are changed by $\Delta\boldsymbol{\phi}$. The system matrix $\mathbf{A}$ will then be affected by $\Delta\mathbf{A}$. We can express

$$\Delta\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{W}^{\mathrm{T}} \tag{1}$$

where $\mathbf{V}$, $\mathbf{D}$ and $\mathbf{W}$ are $n \times r_1$, $r_1 \times r_2$ and $n \times r_2$ matrices, respectively. For a network example, $\mathbf{D}$ can be an $n_\phi \times n_\phi$ diagonal matrix containing variables, and $\mathbf{V}$ and $\mathbf{W}$ are $n \times n_\phi$ matrices containing $+1$ and -1.[5,8]

The effect of $\Delta\boldsymbol{\phi}$ in the response matrix $\mathbf{A}^{-1}$ is defined as

$$\Delta(\mathbf{A}^{-1}) \triangleq (\mathbf{A}+\Delta\mathbf{A})^{-1} - \mathbf{A}^{-1} \tag{2}$$

For the calculation of $\Delta(\mathbf{A}^{-1})$, a commonly suggested method is the Householder formula,[9] which can be represented by

$$\Delta(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}\mathbf{V}(\mathbf{D}^{-1} + \mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{V})^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1} \tag{3}$$

In this formula, $\mathbf{D}$ is required to be a square and non-singular matrix. Even if this can be satisfied, ill-conditioning may still happen when $\mathbf{D}$ is inverted. In fact, cases exist where $\mathbf{D}$ is simply not invertible and additional measures such as the partitioning procedures developed by Hajj[5] and Vlach and Singhal[8] must be applied. Another formula by Householder is[10]

$$\Delta(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}\mathbf{V}\mathbf{D}(\mathbf{D} + \mathbf{D}\mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{V}\mathbf{D})^{-1}\mathbf{D}\mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1} \tag{4}$$

This formula avoids actually performing the inversion of $\mathbf{D}$. But it still has the same limitation as that of (3).

In accordance with the form of $\mathbf{D}$, we refer to (3) as the square with inversion formula (SIF) and (4) as the square without inversion formula (SF).

To alleviate the limitations, corresponding formulae can be derived as[11]

$$\Delta(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}\mathbf{V}\mathbf{D}(\mathbf{1} + \mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{V}\mathbf{D})^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1} \tag{5}$$

and

$$\Delta(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}\mathbf{V}(\mathbf{1} + \mathbf{D}\mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{V})^{-1}\mathbf{D}\mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1} \tag{6}$$

These two formulae permit $\mathbf{D}$ to be singular or even rectangular. Thus, more freedom can be exploited using different formulations of $\mathbf{D}$ and ill-conditioning can be avoided.

The reduced systems in (5) and (6) are of orders r_2 and r_1, respectively, where r_1 is the number of rows of $\mathbf{D}$ and r_2 is the number of columns of $\mathbf{D}$. Therefore, (5) may be preferred if $r_1 > r_2$, otherwise (6) should be used. It is reasonable to refer to (5) as the vertical rectangular formula (VRF) and (6) as the horizontal rectangular formula (HRF), respectively, reflecting the form of $\mathbf{D}$.

The case of a rectangular $\mathbf{D}$ may occur, e.g. when we construct a minimum order reduced system involving variables that are active element parameters, and when large change algorithms are applied to algebraic linear systems other than electrical networks.[11] In those cases, the rectangular $\mathbf{D}$ may be used in the VRF and HRF without modification leaving $\mathbf{V}$ and $\mathbf{W}$ free of values $\Delta\boldsymbol{\phi}$. Hence, $\mathbf{V}$ and $\mathbf{W}$ need to be preprocessed only once.

It should be noted that mathematically, the square formulae are special cases of the rectangular ones. Computationally, the latter have good stability.

PROPERTIES OF THE SET OF GENERALIZED HOUSEHOLDER FORMULAE

Duality property

The HRF and the VRF can be considered as dual to each other. If we apply the following interchanges:

$$\mathbf{A} \leftrightarrow \mathbf{A}^{\mathrm{T}} \tag{7}$$

$$\mathbf{D} \leftrightarrow \mathbf{D}^{\mathrm{T}} \tag{8}$$

and

$$\mathbf{V} \leftrightarrow \mathbf{W} \tag{9}$$

then the two formulae, i.e. (5) and (6), are completely interchanged.

This duality property can be employed to save our analytical effort by half. Unless otherwise stated, we will focus on the vertical formula in the ensuing sections. Results for the horizontal ones can be similarly obtained.

The minimum order of the reduced system

Using the scattering theory approach, Haley has found that the order of the reduced system can be as low as rank ($\Delta\mathbf{A}$). Using our approach of only simple matrix manipulations one can also verify that[11]

$$\min_{(\mathbf{V},\mathbf{D},\mathbf{W})} r_1 = \min_{(\mathbf{V},\mathbf{D},\mathbf{W})} r_2 = \mathrm{rank}\,(\Delta\mathbf{A}) \tag{10}$$

This equation yields the conclusion that, for evaluating large change effects involving Householder formulae, the minimum order of the reduced system is equal to the rank of $\Delta\mathbf{A}$.

Consider the circuit of Figure 1 in which 7 parameters are changed from their nominal values. By the methods of References 3–5 and 8, the reduced system is 7×7. However, the rank of the nodal admittance deviation matrix is 4. Thus, an even smaller system of size 4×4 is sufficient for this problem.

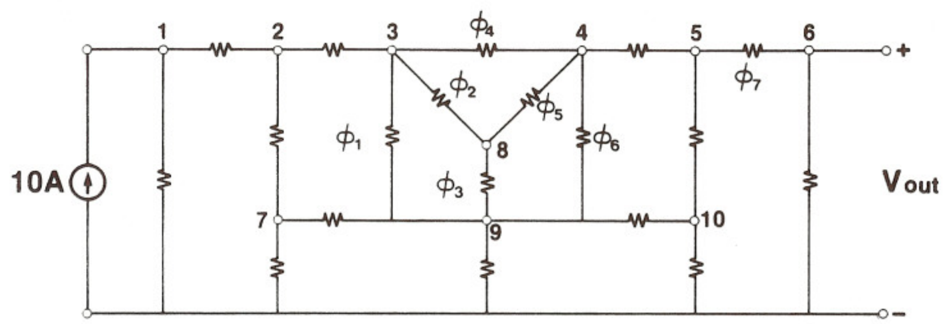

Figure 1. An arbitrary 10 node network with 7 variable parameters. All element values are assumed to be 1. Variables $\phi_1, \phi_2, \ldots, \phi_7$ are conductances of the associated components

Operation count

Consider the computation of $\Delta(\mathbf{A}^{-1})$. Suppose $r_1 + r_2 < n$ and the matrix $\mathbf{A}$ has already been LU factorized. Usually, $\mathbf{V}$, $\mathbf{D}$ and $\mathbf{W}$ are formulated such that $\mathbf{D}$ contains variables whereas $\mathbf{V}$ and $\mathbf{W}$ indicate the positions of the variables and are constant. Preparatory calculations involving $\mathbf{V}$ and $\mathbf{W}$ are performed only once for each set of variables. Table I gives operation counts (number of operations, i.e. multiplications or divisions) for the set of generalized Householder formulae. As shown in the table, the computational stability of the HRF and the VRF is achieved at the cost of one more matrix multiplication, as compared with the SIF. It should be noticed that these operation counts are for arbitrary algebraic linear equations. When linear circuits are concerned, and operation count is reduced as discussed in the next section.

COMPUTATIONS OF ORIGINAL AND ADJOINT LINEAR SYSTEM RESPONSES CORRESPONDING TO DIFFERENT NUMBERS OF INPUTS AND OUTPUTS

In this section, we examine the computations of large change sensitivities in different input and output cases. The VRF is applied. All results of forward and backward substitutions (FBS) involving $\mathbf{A}$ are

Table I. Operation count for the generalized Householder formulae

Cases	Square with inversion formula (SIF)	Vertical rectangular formula (VRF)	Horizontal rectangular formula (HRF)	Square without inversion formula (SF)
Case 1				
$r_1 \neq r_2$				
Preparatory calculation	—	C_P	C_P	—
Calculation for each set of parameter changes	—	C_2	C_1	—
Case 2				
$r_1 = r_2 = r$				
Preparatory calculation	C_P	C_P	C_P	C_P
Calculation for each set of parameter changes	$2C_A + C_B$	$3C_A + C_B$	$3C_A + C_B$	$5C_A + C_B$

$C_P = n^2(r_1 + r_2) + nr_1r_2$
$C_1 = r_1(2r_1r_2 + r_1^2 + r_2n + n^2)$, $C_2 = r_2(2r_1r_2 + r_2^2 + r_1n + n^2)$
$C_A = r^3$, $C_B = rn(r + n)$

calculated in the preparatory step and are represented by $\mathbf{P}$ and $\mathbf{p}$ for the original system (coefficient matrix $\mathbf{A}$) and by $\mathbf{Q}$ and $\mathbf{q}$ for the adjoint system (coefficient matrix $\mathbf{A}^T$). To distinguish these solutions for different right-hand sides, we use characters similar to the right-hand sides as subscripts. For example, $\mathbf{P}_V$ is the solution of

$$\mathbf{A}\mathbf{P}_V = \mathbf{V} \tag{11}$$

and $\mathbf{q}_b$ is the solution of

$$\mathbf{A}^T\mathbf{q}_b = \mathbf{b} \tag{12}$$

Case 1. Response matrix $\mathbf{A}^{-1}$

$$\Delta(\mathbf{A}^{-1}) = -\mathbf{P}_V\mathbf{D}\mathbf{S}\mathbf{Q}_W^T \tag{13}$$

where $\mathbf{S}$ is the inverse of $(\mathbf{1} + \mathbf{W}^T\mathbf{P}_V\mathbf{D})$.

Case 2. System reponses for a single excitation vector $\mathbf{c}$

Suppose that the response vector corresponding to excitation $\mathbf{c}$ is $\mathbf{x} = [x_1\ x_2 \ldots x_n]^T$, i.e.

$$\mathbf{A}\mathbf{x} = \mathbf{c} \tag{14}$$

We have

$$\begin{aligned}\Delta\mathbf{x} &= \Delta(\mathbf{A}^{-1}\mathbf{c})\\ &= -\mathbf{P}_V\mathbf{D}\mathbf{s}\end{aligned} \tag{15}$$

where $\mathbf{s}$ is the solution of

$$(\mathbf{1} + \mathbf{W}^T\mathbf{P}_V\mathbf{D})\mathbf{s} = \mathbf{W}^T\mathbf{x} \tag{16}$$

Case 3. Adjoint responses for a single excitation vector $\mathbf{b}$

Suppose that the adjoint response vector corresponding to excitation $\mathbf{b}$ is $\mathbf{y} = [y_1\ y_2 \ldots y_n]^T$, i.e.

$$\mathbf{A}^T\mathbf{y} = \mathbf{b} \tag{17}$$

We have

$$\Delta \mathbf{y}^{\mathrm{T}} = \Delta(\mathbf{b}^{\mathrm{T}}\mathbf{A}^{-1}) = -\mathbf{s}'^{\mathrm{T}}\mathbf{Q}_{\mathrm{W}}^{\mathrm{T}} \tag{18}$$

where $\mathbf{s}'$ is the solution of

$$(1+\mathbf{Q}_{\mathrm{W}}^{\mathrm{T}}\mathbf{V}\mathbf{D})^{\mathrm{T}}\mathbf{s}' = \mathbf{D}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{q}_{\mathrm{b}} \tag{19}$$

Case 4. Response of a single-input and single-output (SISO) system

If we use vector $\mathbf{b}$ to select the desired output from response vector $\mathbf{x}$, then

$$\Delta(\mathbf{b}^{\mathrm{T}}\mathbf{x}) = \Delta(\mathbf{b}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{c}) = -\mathbf{b}_1^{\mathrm{T}}\mathbf{D}\mathbf{s} \tag{20}$$

where $\mathbf{s}$ is defined in (16) and $\mathbf{b}_1$ equals $\mathbf{P}_{\mathrm{V}}^{\mathrm{T}}\mathbf{b}$ and is obtained in the preparatory step.

Case 5. Responses of a multi-input and multi-output (MIMO) system

Suppose $\mathbf{C}$ is an $n \times n'$ matrix whose columns represent different excitation vectors and $\mathbf{B}$ is an $n \times m'$ matrix whose columns select the desired output measurements. Then the n'-input-m'-output case can be expressed, formally, by $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{C}$. Thus

$$\Delta(\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{C}) = -\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{V}\mathbf{D}(1+\mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{V}\mathbf{D})^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{C} \tag{21}$$

We notice that the term $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{V}$ can be computed either as $\mathbf{B}^{\mathrm{T}}\mathbf{P}_{\mathrm{V}}$ or $\mathbf{Q}_{\mathrm{B}}^{\mathrm{T}}\mathbf{V}$ with the difference of operation count being $n^2(r_1 - m')$. Therefore, comparing r_1 and m', we can calculate $(\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{V})$ as

$$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{V} = \begin{cases} \mathbf{B}^{\mathrm{T}}\mathbf{P}_{\mathrm{V}} & \text{if } r_1 \leq m' \quad \text{(22a)} \\ \mathbf{Q}_{\mathrm{B}}^{\mathrm{T}}\mathbf{V} & \text{if } r_1 > m' \quad \text{(22b)} \end{cases}$$

Similarly,

$$\mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{C} = \begin{cases} \mathbf{W}^{\mathrm{T}}\mathbf{P}_{\mathrm{C}} & \text{if } r_2 > n' \quad \text{(23a)} \\ \mathbf{Q}_{\mathrm{W}}^{\mathrm{T}}\mathbf{C} & \text{if } r_2 \leq n' \quad \text{(23b)} \end{cases}$$

Also, at least one of (22a) and (23b) should be used in order to yield either $\mathbf{P}_{\mathrm{V}}$ or $\mathbf{Q}_{\mathrm{W}}$ which is required in calculating

$$(1+\mathbf{W}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{V}\mathbf{D}) = (1+\mathbf{Q}_{\mathrm{W}}^{\mathrm{T}}\mathbf{V}\mathbf{D}) = (1+\mathbf{W}^{\mathrm{T}}\mathbf{P}_{\mathrm{V}}\mathbf{D}) \tag{24}$$

Hence, according to the values of r_1, r_2, m' and n', we can choose appropriate formulations. For example, when $m' < n'$ and $m' < r_2$, we use

$$\Delta(\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{C}) = -\mathbf{S}^{\mathrm{T}}\mathbf{Q}_{\mathrm{W}}^{\mathrm{T}}\mathbf{C} \tag{25}$$

where $\mathbf{S}$ is the solution to

$$(1+\mathbf{Q}_{\mathrm{W}}^{\mathrm{T}}\mathbf{V}\mathbf{D})^{\mathrm{T}}\mathbf{S} = (\mathbf{Q}_{\mathrm{B}}^{\mathrm{T}}\mathbf{V}\mathbf{D})^{\mathrm{T}} \tag{26}$$

This approach requires $m' + r_2$ FBS in the adjoint system for $\mathbf{Q}_{\mathrm{B}}$ and $\mathbf{Q}_{\mathrm{W}}$ as preparatory calculations, one LU factorization and m' FBS in the reduced system of (26).

Expressions for different cases of large change evaluation

In Table II, we summarize the various cases of the above discussion. Different situations of the MIMO case are distinguished so that the number of FBS in the $n \times n$ system equals the minimum of $m' + r_2$,

Table II. Formulae for the computation of large changes when $\mathbf{A}^{-1}$ is involved and when $r_1 \geq r_2$

Identification	Formula	Definition of S or s*
$\Delta(\mathbf{A}^{-1})$	$-\mathbf{P}_V\mathbf{D}\mathbf{S}\mathbf{Q}_W^T$	$\mathbf{H}_1\mathbf{S}=\mathbf{1}$ or $\mathbf{H}_2\mathbf{S}=\mathbf{1}$
$\Delta(\mathbf{b}^T\mathbf{A}^{-1})$	$-\mathbf{s}^T\mathbf{Q}_W^T$	$\mathbf{H}_1^T\mathbf{s}=\mathbf{D}^T(\mathbf{V}^T\mathbf{q}_b)$
$\Delta(\mathbf{A}^{-1}\mathbf{c})$	$-\mathbf{P}_V\mathbf{D}\mathbf{s}$	$\mathbf{H}_2\mathbf{s}=\mathbf{W}^T\mathbf{p}_c$
$\Delta(\mathbf{b}^T\mathbf{A}^{-1}\mathbf{c})$	$-(\mathbf{b}^T\mathbf{P}_V)\mathbf{D}\mathbf{s}$	$\mathbf{H}_2\mathbf{s}=\mathbf{W}^T\mathbf{p}_c$
†$\Delta(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{C})$	(1) $-\mathbf{S}^T(\mathbf{Q}_W^T\mathbf{C})$	$\mathbf{H}_1^T\mathbf{S}=\mathbf{D}^T(\mathbf{V}^T\mathbf{Q}_B)$
	(2) $-(\mathbf{B}^T\mathbf{P}_V)\mathbf{D}\mathbf{S}$	$\mathbf{H}_2\mathbf{S}=\mathbf{W}^T\mathbf{P}_C$
	(3) $-(\mathbf{Q}_B^T\mathbf{V})\mathbf{D}\mathbf{S}$	$\mathbf{H}_1\mathbf{S}=\mathbf{Q}_W^T\mathbf{C}$
	(4) $-(\mathbf{B}^T\mathbf{P}_V)\mathbf{D}\mathbf{S}(\mathbf{Q}_W^T\mathbf{C})$	$\mathbf{H}_1\mathbf{S}=\mathbf{1}$ or $\mathbf{H}_2\mathbf{S}=\mathbf{1}$
	(5) $-(\mathbf{Q}_B^T\mathbf{V})\mathbf{D}\mathbf{S}(\mathbf{Q}_W^T\mathbf{C})$	$\mathbf{H}_1\mathbf{S}=\mathbf{1}$

* $\mathbf{H}_1=(\mathbf{1}+\mathbf{Q}_W^T\mathbf{V}\mathbf{D})$, $\mathbf{H}_2=(\mathbf{1}+\mathbf{W}^T\mathbf{P}_V\mathbf{D})$

† Table III can be used as a guide to select among (1) to (5) by the minimum FBS criterion.

$n'+r_1$ and r_1+r_2 and the number of FBS in the reduced system equals the minimum of r_1, r_2, m' and n', as shown in Table III. This minimum FBS criterion can be used as a guide to select appropriate expressions for the calculation of $\Delta(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{C})$.

When the number of FBS exceeds the order of the system, a matrix inversion may be performed directly.

Computational cost consideration

In the preceding section the operation count was discussed for a general linear system of equations. However, when an electric circuit is concerned, the cost is much less. We consider the SISO network as an example. Suppose that the reduced system is of order r. In the preparatory step, we calculate $\mathbf{P}_V$, whose operation count is rn^2, and $\mathbf{P}_V^T\mathbf{b}$, $\mathbf{W}^T\mathbf{P}_V$ and $\mathbf{W}^T\mathbf{x}$ which are simply element selections and additions. Then, for each set of parameter changes, we formulate and solve the reduced system by at worst $4r^3/3 - r/3 + r^2$ operations. The operation count for updating the output is r for the SIF and $r+r^2$ for the HRF and the VRF.

Special case: first-order sensitivity

As a special case of large change sensitivity analysis, small change sensitivity computations can be deduced from our large change formulae without reference to Tellegen's theorem. Table IV gives examples of such first-order sensitivities w.r.t. components of a matrix. Table V lists formulae w.r.t. variables. These results are obtained by putting $\Delta\phi$ into the denominators of large change formulae and then letting the

Table III. Major computational effort for calculating $\Delta(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{C})$ by formulae in Table II where $r_1 \geq r_2$

Category	Corresponding Case in Table II	The $n \times n$ system represented by $\mathbf{A}$	The $r_2 \times r_2$ system represented by $\mathbf{H}_1$ or $\mathbf{H}_2$
No. of LU factorizations	(1)-(5)	1	1
No. of FBS	(1)	$m'+r_2$	m'
	(2)	$n'+r_1$	n'
	(3)	$m'+r_2$	n'
	(4)	r_1+r_2	r_2
	(5)	$m'+r_2$	r_2

Table IV. Expressions appropriate for computations of sensitivitites w.r.t. components of matrix $\mathbf{A}$ when $\mathbf{A}^{-1}$ is involved

Identification	Sensitivity expression*	
	(a) general	(b) when $\mathbf{A}=\mathbf{A}^{\mathrm{T}}$ and $i \neq j$
$\dfrac{\partial \mathbf{A}^{-1}}{\partial A_{ij}}$	$-\mathbf{p}_{\mathrm{u}i}\mathbf{q}_{\mathrm{u}j}^{\mathrm{T}}$	$-(\mathbf{p}_{\mathrm{u}i}\mathbf{p}_{\mathrm{u}j}^{\mathrm{T}}+\mathbf{p}_{\mathrm{u}j}\mathbf{p}_{\mathrm{u}i}^{\mathrm{T}})$
$\dfrac{\partial(\mathbf{b}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{c})}{\partial \mathbf{A}}$	$-\mathbf{q}_{\mathrm{b}}\mathbf{p}_{\mathrm{c}}^{\mathrm{T}}$	$-(\mathbf{p}_{\mathrm{b}}\mathbf{p}_{\mathrm{c}}^{\mathrm{T}}+\mathbf{p}_{\mathrm{c}}\mathbf{p}_{\mathrm{b}}^{\mathrm{T}})$
$\dfrac{\partial(\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{C})}{\partial A_{ij}}$	$-\mathbf{B}^{\mathrm{T}}\mathbf{p}_{\mathrm{u}i}\mathbf{q}_{\mathrm{u}j}^{\mathrm{T}}\mathbf{C}$	$-\mathbf{B}^{\mathrm{T}}(\mathbf{p}_{\mathrm{u}i}\mathbf{p}_{\mathrm{u}j}^{\mathrm{T}}+\mathbf{p}_{\mathrm{u}j}\mathbf{p}_{\mathrm{u}i}^{\mathrm{T}})\mathbf{C}$
$\dfrac{\partial[\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{C}]_{lk}}{\partial \mathbf{A}}$ †	$-\mathbf{q}_{\mathrm{b}}\mathbf{p}_{\mathrm{c}}^{\mathrm{T}}$	$-(\mathbf{p}_{\mathrm{b}}\mathbf{p}_{\mathrm{c}}^{\mathrm{T}}+\mathbf{p}_{\mathrm{c}}\mathbf{p}_{\mathrm{b}}^{\mathrm{T}})$‡

* $\mathbf{u}_i(\mathbf{u}_j)$ is a unit n-vector containing 1 at the ith (jth) row and zeros everywhere else.

† $[*]_{lk}$ is the (l, k)th element of matrix $*$.

‡ $\mathbf{b}$ is the lth column of $\mathbf{B}$ and $\mathbf{c}$ is the kth column of $\mathbf{C}$. Both $\mathbf{b}$ and $\mathbf{c}$ are used as the R.H.S. of the system involving $\mathbf{A}$ for original solutions $\mathbf{p}_{\mathrm{b}}$, $\mathbf{p}_{\mathrm{c}}$ and adjoint solution $\mathbf{q}_{\mathrm{b}}$.

parameter change $\Delta\phi$ approach zero. The formulae in Tables IV and V are consistent with the existing ones derived using other approaches, e.g. Reference 12.

EXAMPLES

Example 1. A system of linear equations with rectangular **D**

Consider a 10×10 system of linear equations with coefficient matrix $\mathbf{A}$. Suppose the intersection elements of rows 2, 5, 9 and columns 3 and 6 are constantly changed. We formulate $\mathbf{V}$, $\mathbf{D}$ and $\mathbf{W}$ such that

$$\mathbf{V}=[\mathbf{u}_2\ \mathbf{u}_5\ \mathbf{u}_9] \tag{27}$$

$$\mathbf{W}=[\mathbf{u}_3\ \mathbf{u}_6] \tag{28}$$

and

$$\mathbf{D}=\begin{bmatrix} \Delta A_{23} & \Delta A_{26} \\ \Delta A_{53} & \Delta A_{56} \\ \Delta A_{93} & \Delta A_{96} \end{bmatrix}, \tag{29}$$

where $\mathbf{u}_i$, $i=2, 3, 5, 6, 9$, is a unit 10-vector containing 1 in the ith row and zeros everywhere else. In this way, no additional effort is involved when applying the VRF and HRF. If we use the square formulae, elementary transformations must be employed in order to obtain a square matrix $\mathbf{D}$.

Numerical solutions as well as intermediate results are shown in Figure 2.

Example 2. An electrical network with its minimum order system achieved

The 10-node circuit of Figure 1 is solved using the generalized Householder formulae with simultaneous changes of 7 variable components. The minimum order of the reduced system is 4, which is achieved by

Table V. Expressions appropriate for computation of sensitivities w.r.t. variable ϕ when $\mathbf{A}^{-1}$ is involved

Identification	Sensitivity expression
$\dfrac{\partial \mathbf{A}^{-1}}{\partial \phi}$	$-\mathbf{P}_{\mathrm{U}_I}\dfrac{\partial \mathbf{A}_{IJ}}{\partial \phi}\mathbf{Q}^{\mathrm{T}}_{\mathrm{U}_J}$
$\dfrac{\partial (\mathbf{b}^{\mathrm{T}}\mathbf{A}^{-1})}{\partial \phi}$	$-(\mathbf{q}^{\mathrm{T}}_{\mathrm{b}})_I\dfrac{\partial \mathbf{A}_{IJ}}{\partial \phi}\mathbf{Q}^{\mathrm{T}}_{\mathrm{U}_J}$
$\dfrac{\partial (\mathbf{A}^{-1}\mathbf{c})}{\partial \phi}$	$-\mathbf{P}_{\mathrm{U}_I}\dfrac{\partial \mathbf{A}_{IJ}}{\partial \phi}(\mathbf{p}_{\mathrm{c}})_J$
$\dfrac{\partial (\mathbf{b}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{c})}{\partial \phi}$	$-(\mathbf{q}^{\mathrm{T}}_{\mathrm{b}})_I\dfrac{\partial \mathbf{A}_{IJ}}{\partial \phi}(\mathbf{p}_{\mathrm{c}})_J$
$\dfrac{\partial (\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{C})}{\partial \phi}$	$-(*)\dfrac{\partial \mathbf{A}_{IJ}}{\partial \phi}(\dagger)$

$I(J)$ is an index set whose elements indicate the rows (columns) containing the variable ϕ.

$\mathbf{A}_{IJ}$ is a matrix containing the intersection elements of $\mathbf{A}$ in rows i, $i \in I$ and columns j, $j \in J$.

$\mathbf{U}_I(\mathbf{U}_J)$ is a matrix whose columns are unit vectors $\mathbf{u}_i$, $i \in I(\mathbf{u}_j, j \in J)$.

$$(*) = \begin{cases} \mathbf{B}^{\mathrm{T}}\mathbf{P}_{\mathrm{U}_I}, & \text{if } n_I < m', \\ \mathbf{Q}^{\mathrm{T}}_{\mathrm{B}}\mathbf{U}_I, & \text{if } n_I \geqslant m' \end{cases} \qquad (\dagger) = \begin{cases} \mathbf{Q}^{\mathrm{T}}_{\mathrm{U}_J}\mathbf{C}, & \text{if } n_J < n' \\ \mathbf{U}^{\mathrm{T}}_J\mathbf{P}_{\mathrm{C}}, & \text{if } n_J \geqslant n' \end{cases}$$

$(\mathbf{q}_{\mathrm{b}})_I$ and $(\mathbf{p}_{\mathrm{c}})_J$ are defined as vectors consisting of all ith elements of $\mathbf{q}_{\mathrm{b}}$, $i \in I$, and all jth elements of $\mathbf{p}_{\mathrm{c}}$, $j \in J$, respectively.

formulating **V**, **D** and **W** as

$$\mathbf{V} = \mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

and

$$\mathbf{D} = \begin{bmatrix} \Delta\phi_1 + \Delta\phi_2 + \Delta\phi_4 & -\Delta\phi_4 & -\Delta\phi_2 & 0 \\ -\Delta\phi_4 & \Delta\phi_4 + \Delta\phi_5 + \Delta\phi_6 & -\Delta\phi_5 & 0 \\ -\Delta\phi_2 & -\Delta\phi_5 & \Delta\phi_2 + \Delta\phi_3 + \Delta\phi_5 & 0 \\ 0 & 0 & 0 & \Delta\phi_7 \end{bmatrix}. \tag{31}$$

The -1's in **V** and **W** correspond to reference nodes associated with the variables. If a loop or several connected loops are fomulated by the variable branches, a common reference node is appointed for all the variables contributed to the loop or loops. For example, node 9 in Figure 1 is chosen as the common

MATRIX [A]										VECTOR [B]
1.0	5.0	5.0	1.0	5.0	2.0	1.0	1.0	7.0	2.0	35.0
2.0	3.0	3.0	7.0	0.0	4.0	3.0	6.0	8.0	3.0	32.0
3.0	0.0	2.0	4.0	2.0	6.0	4.0	4.0	9.0	7.0	16.0
6.0	1.0	2.0	5.0	2.0	3.0	3.0	7.0	3.0	5.0	51.0
8.0	1.0	2.0	2.0	4.0	4.0	6.0	8.0	4.0	8.0	42.0
4.0	1.0	6.0	7.0	3.0	5.0	7.0	3.0	5.0	3.0	19.0
7.0	0.0	6.0	5.0	9.0	4.0	8.0	9.0	2.0	[illegible].0	34.0
2.0	0.0	4.0	2.0	2.0	5.0	3.0	5.0	4.0	3.0	71.0
3.0	2.0	0.0	1.0	5.0	3.0	4.0	2.0	3.0	1.0	36.0
4.0	2.0	4.0	4.0	6.0	2.0	9.0	6.0	1.0	7.0	61.0

SOLUTION BEFORE ANY CHANGE :

VECTOR [X]
-8.89217
39.80097
-3.00067
2.31014
-5.40544
48.42778
-12.11626
-3.61726
-32.93004
16.99799

Figure 2(a). The original linear system and its solutions. **A** is a 10×10 matrix containing parameters of the system; **b** is the excitation vector; **x** is the solution vector

reference node for variables $\phi_1, \phi_2, \ldots, \phi_6$ and -1 appears in the 9th row of **V** accordingly. The 1's in and **W** correspond to the non-reference nodes associated with the variable branches, e.g. nodes 3, 4, and 8 in Figure 1. With respect to each reference node, a submatrix is formulated using $\Delta\phi$ in just the same way as if a nodal admittance matrix is formulated using ϕ w.r.t. a ground node. **D** is a block diagonal matrix containing those submatrices.

The changes of variables range from 0·00001 to 90. Zero changes are also included, as shown in Table VI. These simultaneous small, large and zero changes are handled directly by the VRF.[11] For the two extreme cases of $\Delta\phi$, the SIF can handle $\Delta\phi \to \infty$ whereas the VRF and HRF accommodate $\Delta\phi \to 0$. In a Monte Carlo analysis, network optimization, identification and tuning, various unpredictable patterns of $\Delta\phi \to 0$ in multiparameter changes may be possible, and $\Delta\phi \to \infty$ is often limited by, for instance, tolerances and tuning ranges or by step size constraints. For 100 sets of variable changes of ϕ_1 to ϕ_7, the operation

MATRIX [V]

0.0	0.0	0.0
1.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	1.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	1.0
0.0	0.0	0.0

MATRIX [W]

0.0	0.0
0.0	0.0
1.0	0.0
0.0	0.0
0.0	0.0
0.0	1.0
0.0	0.0
0.0	0.0
0.0	0.0
0.0	0.0

MATRIX [PV]

-.03684	.19072	-.00936
-.30799	-.26526	.04838
.00454	.09406	-.15865
-.04645	-.23600	.02949
.02608	-.09579	.12002
-.48948	-.43199	.13012
.18919	.22984	.01487
.27060	.13754	.00846
.36321	.32717	-.02238
-.27658	-.20670	-.09789

VECTOR [RHS]

-3.00067
48.42778

Figure 2(b). Matrices **V**, **W**, $\mathbf{P}_V$ and vector **RHS**, where $\mathbf{P}_V$ is the solution of $\mathbf{AP}_V = \mathbf{V}$ and $\mathbf{RHS} = \mathbf{W}^T\mathbf{x}$

counts for our method using SIF, VRF, the existing method of References 3–5 and 8 and the direct method are of the order of 11,230 14,030, 24,930 and 43,430, respectively.

CONCLUSIONS

We have presented a multiparameter large change sensitivity analysis approach for a general system involving solutions of linear equations. Particular attention has been devoted to the formulation and order

MATRIX [D]

2.00000	3.00000
4.00000	5.00000
2.00000	3.00000

MATRIX [H]

1.06802	.00797
-2.44666	-2.23801

VECTOR [S]

-2.66983

-18.72004

SOLUTION AFTER THE FIRST LARGE CHANGE :

VECTOR [X]

8.15496

-3.82546

-2.66983

-23.34277

-6.40995

-18.72004

24.40133

27.88727

22.14824

-27.58607

Figure 2(c). Results corresponding to the first change of variable parameters represented by **D**. **H** represents $(\mathbf{1}+\mathbf{W}^{T}\mathbf{A}^{-1}\mathbf{V}\mathbf{D})$ and **s** is the solution of the reduced system $\mathbf{Hs}=\mathbf{W}^{T}\mathbf{x}$

of the reduced system, which in turn affects the stability and efficiency of the system repsonse evaluation. The mathematical essence of the generalized Householder formulae also provides basic links with other approaches, indicating their theoretical equivalence. However, our extended formulae accommodate more cases of various formulations of the reduced system which the traditional methods cannot handle directly. For a general circuit with arbitrary distribution of variable components, proper formulations of **V**, **D** and **W** are possible to ensure that the large change calculation is performed via a minimum order reduced system. Thus, under certain circumstances, large change algorithms are still feasible even if many system

```
VARIABLES CHANGE AGAIN CAUSING A CHANGE OF [D].
[V] AND [W] REMAIN UNCHANGED.

                    MATRIX [D]
            6.00000         7.00000
            5.00000         4.00000
            3.00000         4.00000


                    MATRIX [H]
            1.02160         -.22657
           -4.70646        -3.63383


                    VECTOR [S]
                    -4.57788
                    -7.39775


      SOLUTION AFTER THE SECOND LARGE CHANGE :

                    VECTOR [X]
                    -2.20815
                     3.56798
                    -4.57788
                   -12.47901
                    -3.16642
                    -7.39775
                    15.58395
                    25.41279
                    12.05534
                   -20.00992
```

Figure 2(d). Results corresponding to the second change of variable parameters. **H** and **s** are similarly defined to those in Figure 2(c)

parameters are changed. These circumstances may be, for example, a case where loops are constructed by branches containing variables. It is also possible that a general formulation of **V**, **D** and **W**, together with the set of Householder formulae, can be embedded into the different iterative and non-iterative methods of Hajj[5] to yield various powerful design procedures.

Table VI. Parameter changes for example 2

Variable	The first change $(1/\Omega)$	The second change $(1/\Omega)$	The third change $(1/\Omega)$
$\Delta\phi_1$	84·0	0·00001	0·2
$\Delta\phi_2$	0·5	0·001	0
$\Delta\phi_3$	0·00001	0·12	3·0
$\Delta\phi_4$	0·02	45	0
$\Delta\phi_5$	40	0·00003	0·02
$\Delta\phi_6$	50	90	15
$\Delta\phi_7$	0·00002	−2	0·1

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