Brief description for incorporating a circuit element into the admittance matrix (Y matrix) of the overall circuit, and related adjoint sensitivity.

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Notations

Suppose the overall circuit has N nodes. Suppose **V** is a vector of size N containing node voltages of the circuit. Suppose **I** is a vector of size N containing independent current sources of the circuit. Let **Y** be a N by N matrix representing the nodal admittance matrix of the entire circuit.

The elements of the **V** vector are:

$$\mathbf{V} = [V_1 \ \ V_2 \ \ldots \ V_N]^T$$

Where V_i, i = 1, 2, …, N, is the nodal voltage at node i.

Circuit Simulation:

Given **Y** and **I**, the **V** vector can be solved by solving the linear equations:

$$\mathbf{Y}\,\mathbf{V} = \mathbf{I}$$

Adjoint Voltages

Let V_{out} be the output voltage of interests in the circuit. V_{out} can be expressed as

$$V_{out} = \mathbf{u}^T\mathbf{V}$$

where **u** is a constant vector of size N. For example, if V_{out} is the nodal voltage at node 2 of the circuit, **u** will be a vector with a value of 1 at the 2[nd] location and zero everywhere else.

Let $\hat{\mathbf{V}}$ be a vector of size N containing adjoint voltages

$$\hat{\mathbf{V}} = [\hat{V}_1 \ \ \hat{V}_2 \ \ldots \ \hat{V}_N]^T$$

$\hat{\mathbf{V}}$ can be solved by the following linear equations:

$$\mathbf{Y}^T\,\hat{\mathbf{V}} = \mathbf{u}$$

Example 1: Resistor

Suppose that a resistor is connected between node i and node j.

The contribution of the resistor to the Y matrix of the overall circuit is zero everywhere except 4 locations in the Y matrix:

$$
\begin{array}{c c}
\text{rows } i \rightarrow & \\
\text{row } j \rightarrow & \\
\end{array}
\begin{bmatrix} \frac{1}{R} & -\frac{1}{R} \\ -\frac{1}{R} & \frac{1}{R} \end{bmatrix}_{N \times N}
$$
$$
\begin{array}{cc}
\uparrow & \uparrow \\
\text{column } i & \text{column } j
\end{array}
$$

The sensitivity of the V_{out} with respect to parameter R is:

$$
\frac{\partial V_{out}}{\partial R} = -(\hat{V}_i - \hat{V}_j)\frac{\partial(\frac{1}{R})}{\partial R}(V_i - V_j)
$$
$$
= -(\hat{V}_i - \hat{V}_j)(-\frac{1}{R^2})(V_i - V_j)
$$

Example 2: Capacitor

Suppose that a capacitor is connected between node i and node j.

The contribution of the capacitor to the Y matrix of the overall circuit is zero everywhere except 4 locations in the Y matrix:

$$\begin{array}{r c}
 & \begin{array}{cc} \text{column } i & \text{column } j \end{array} \\
\begin{array}{r} \text{row } i \rightarrow \\ \text{row } j \rightarrow \end{array} & \begin{bmatrix} j\omega C & -j\omega C \\ -j\omega C & j\omega C \end{bmatrix}_{N \times N}
\end{array}$$

The sensitivity of the V_{out} with respect to parameter C is:

$$\frac{\partial V_{out}}{\partial C} = -(\hat{V}_i - \hat{V}_j)\,\frac{\partial(j\omega C)}{\partial C}\,(V_i - V_j)$$
$$= -(\hat{V}_i - \hat{V}_j)(j\omega)(V_i - V_j)$$

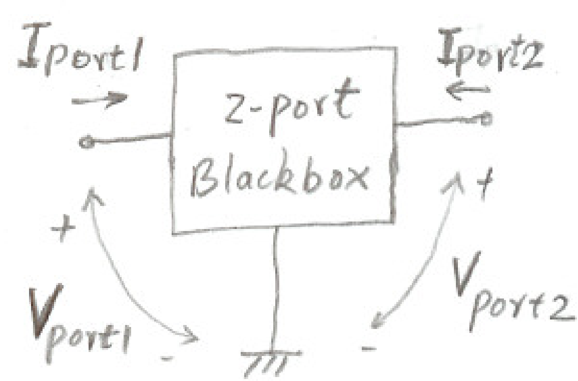
Example 3: A 2-Port Blackbox Element

Suppose that a 2-port blackbox element is represented by its short-circuit admittance matrix representation. Let $\mathbf{Y}_{Port}$ be such a matrix:

$$\underset{\sim}{Y}_{port} = \begin{bmatrix} Y_{port\,11} & Y_{port\,12} \\ Y_{port\,21} & Y_{port\,22} \end{bmatrix}$$

Where

$$Y_{port\,11} = \left.\frac{I_{port1}}{V_{port1}}\right|_{V_{port2}=0}$$

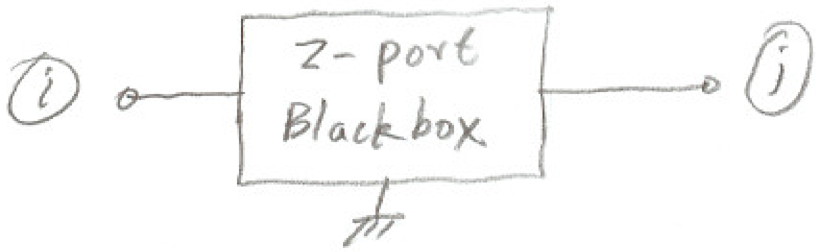


$$Y_{port\,21} = \left.\frac{I_{port2}}{V_{port1}}\right|_{V_{port2}=0}$$

$$Y_{port12} = \left.\frac{I_{port1}}{V_{port2}}\right|_{V_{port1}=0}$$

$$Y_{port22} = \left.\frac{I_{port2}}{V_{port2}}\right|_{V_{port1}=0}$$

Suppose this 2-port element is connected to the overall circuit this way: port 1 of the blackbox is connected to node i of the circuit, and port 2 of the blackbox is connected to node j of the circuit.



The contribution of the 2-port blackbox to the Y matrix of the overall circuit is zero everywhere except 4 locations in the Y matrix of the overall circuit:

$$\begin{array}{rc} \begin{array}{r} \text{row } i \rightarrow \\ \\ \text{row } j \rightarrow \end{array} & \begin{bmatrix} Y_{port11} & Y_{port12} \\ \\ Y_{port21} & Y_{port22} \end{bmatrix} \\ & \begin{array}{cc} \uparrow & \uparrow \\ \text{column } i & \text{column } j \end{array} \end{array}$$

The sensitivity of the V_{out} with respect to any parameter inside the 2-port blackbox is:

$$\frac{\partial V_{out}}{\partial \phi} = -\left[\, \hat{v}_i \quad \hat{v}_j \,\right] \frac{\partial \underset{\sim}{Y}_{port}}{\partial \phi} \begin{bmatrix} v_i \\ v_j \end{bmatrix}$$

where ϕ is a generic variable inside the 2-port blackbox

Example 4: 2-Port Blackbox Originally represented by S-parameters

If the original representation of the 2-port is by other than the admittance (Y-parameter) parameters, we can first convert them into admittance parameter and then follow Example 3 to incorporate the admittance parameters into the overall circuit matrix.

Examples of 2-port parameters: S-parameters, Z-parameters, ABCD matrix. They can all be converted into Y-parameters using standard formulas in textbooks. Attached is a table of such formulas from the book by D.M. Pozar, Microwave Engineering, Addison-Wesley, Reading, MA, 1990, pp 235.

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TABLE 5.2 Conversions Between Two-Port Network Parameters

	S	Z	Y	$ABCD$
S_{11}	S_{11}	$\frac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0-Y_{11})(Y_0+Y_0)+Y_{12}Y_{21}}{\Delta Y}$	$\frac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\frac{2Z_{12}Z_0}{\Delta Z}$	$\frac{-2Y_{12}Y_0}{\Delta Y}$	$\frac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\frac{2Z_{21}Z_0}{\Delta Z}$	$\frac{-2Y_{21}Y_0}{\Delta Y}$	$\frac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\frac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\frac{Y_{22}}{\|Y\|}$	$\frac{A}{C}$
Z_{12}	$Z_0\frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\frac{-Y_{12}}{\|Y\|}$	$\frac{AD-BC}{C}$
Z_{21}	$Z_0\frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\frac{-Y_{21}}{\|Y\|}$	$\frac{1}{C}$
Z_{22}	$Z_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\frac{Y_{11}}{\|Y\|}$	$\frac{D}{C}$
Y_{11}	$Y_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{22}}{\|Z\|}$	Y_{11}	$\frac{D}{B}$
Y_{12}	$Y_0\frac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{12}}{\|Z\|}$	Y_{12}	$\frac{BC-AD}{B}$
Y_{21}	$Y_0\frac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{21}}{\|Z\|}$	Y_{21}	$\frac{-1}{B}$
Y_{22}	$Y_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{11}}{\|Z\|}$	Y_{22}	$\frac{A}{B}$
A	$\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{11}}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\frac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{\|Z\|}{Z_{21}}$	$\frac{-1}{Y_{21}}$	B
C	$\frac{1}{Z_0}\frac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{1}{Z_{21}}$	$\frac{-\|Y\|}{Y_{21}}$	C
D	$\frac{(1-S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$\frac{-Y_{11}}{Y_{21}}$	D

$|Z| = Z_{11}Z_{22} - Z_{12}Z_{21}$; $|Y| = Y_{11}Y_{22} - Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0) - Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0) - Z_{12}Z_{21}$; $Y_0 = 1/Z_0$

Exercise 1: Implementation

Attached file "UserCKTComponent_1.cpp" is a c++ program implementation of Example 1 (the resistor example). As an exercise, you can use this file as a template to do an implementation of Example 2 (the capacitor example).

The symbols used in the c++ program correspond to our notations according to following table.

Symbols in C++ program	Notation used in Theory	Explanation
MNA_Matrix[][]	$\mathbf{Y}$ matrix	nodal admittance matrix of the overall circuit
MNA_V[]	$\mathbf{V}$ vector	vector of nodal voltages
MNA_VHat[]	$\hat{\mathbf{V}}$ vector	vector of adjoint nodal voltages
omega	ω	angular frequency

Note: the word "MNA" precisely mean "Modified Nodal Admittance". In all our examples (Examples 1 to 4), there is no "modification" necessary therefore MNA matrix is the nodal admittance matrix.