Question 1

Let $\phi = [\phi_1, \phi_2]^T$ be a vector of design variables. Error functions for design optimization are

$$
\begin{align*}
  f_1(\phi) &= \phi_2 - 4 \\
  f_2(\phi) &= 2 - \phi_2 \\
  f_3(\phi) &= 1 - \phi_1 \\
  f_4(\phi) &= \phi_1 - 7
\end{align*}
$$

Let $\phi^0$ be the nominal point (design center) and $x$ be a 2-dimensional random variable. We consider 5 random outcomes of $x$:

$$
\begin{align*}
  x^{(1)} &= [0, 2]^T, & x^{(2)} &= [-2, 0]^T, & x^{(3)} &= [-1, -2]^T, \\
  x^{(4)} &= [1, -2]^T, & x^{(5)} &= [2, 0]^T.
\end{align*}
$$

The corresponding 5 circuit outcomes are

$$
\phi^{(k)} = \phi^0 + x^{(k)}, \quad k = 1, 2, \ldots, 5.
$$

The objective function for generalized $L_p$ centering is

$$
U(\phi^0) = H_p(u)
$$

where the kth component of vector $u$ is defined as

$$
u_k = \alpha_k H_q(f(\phi^{(k)})), \quad k = 1, 2, \ldots, 5$$

and the weighting factor $\alpha_k = 1$ for $k = 1, 2, \ldots, 5$.

Suppose the starting point for optimization is $\phi^0 = [\phi^0_1, \phi^0_2]^T = [4.0, 4.2]^T$. Suppose $\phi^0_1$ is fixed at 4.0 throughout optimization.

Case 1: $p = q = 1$

Requirements:

1. Derive the expressions for the objective function $U$ vs. variable $\phi^0_2$ for $0 \leq \phi^0_2 < \infty$. Draw graphically $U$ vs. $\phi^0_2$ for $0 \leq \phi^0_2 < \infty$.
2. Find the optimal design center $\phi^0$, i.e., the minimum of $U(\phi^0)$.
3. Draw the 2-dimensional acceptance region and the locations of the 5 circuit outcomes before optimization.
4. Draw the 2-dimensional acceptance region and the locations of the 5 circuit outcomes after optimization.

Case 2: $p = q = \infty$

Requirements (1) - (4): same as in Case 1.

Compare the yield and worst-case performances between the solutions of the two cases.
Solution

Case 1

(1) Since $\phi_1^o$ is fixed, $U(\phi^o)$ is a function of only $\phi_2^o$.

$$\phi''_2 = \phi^o + \chi'' = \begin{bmatrix} 4 \\ \phi_2^o + 2 \end{bmatrix}$$

$$f_1(\phi^{"}_2) = \phi_2^o - 2 = \begin{cases} > 0 & \text{if } \phi_2^o > 2 \\ \leq 0 & \text{if } \phi_2^o \leq 2 \end{cases}$$

$$f_2(\phi^{"}_2) = -\phi_2^o = \begin{cases} > 0 & \text{if } \phi_2^o < 0 \\ \leq 0 & \text{if } \phi_2^o \geq 0 \end{cases}$$

$$f_3(\phi^{"}_2) = -3 \leq 0$$

$$f_4(\phi^{"}_2) = -3 \leq 0$$

$$u_1 = \left\{ \begin{array}{ll} \sum_{j \in J} f_j(\phi^{"}_2) &= f_2(\phi^{"}_2) = -\phi_2^o, & \text{if } \phi_2^o \leq 0 \quad (J = \{2\}) \\ \sum_{j \in J} f_j(\phi^{"}_2) &= f_1(\phi^{"}_2) = \phi_2^o - 2, & \text{if } \phi_2^o > 2 \quad (J = \{1\}) \\ -\left(\sum_{j=1}^{4} (-f_j(\phi^{"}_2))^{-1}\right)^{-1} & , & \text{if } 0 < \phi_2^o < 2 \\ \end{array} \right.$$
Similarly:

\[ u_2 = H_1 \left( \frac{1}{2} \phi^{(2)} \right) = \begin{cases} \frac{1}{2} \phi^{(2)} = \phi_2^0 - 4 & \text{if } \phi_2^0 \geq 4 \\ \phi_2^0 = 2 - \phi_2^0 & \text{if } \phi_2^0 \\ -\left( \sum_{j=1}^{n} \left( -f_j \phi^{(2)} \right) \right)^{-1} & \text{if } 2 < \phi_2^0 \leq 4 \end{cases} \]

Notice \( u_2 = \begin{cases} > 0 & \text{if } \phi_2^0 < 2 \text{ or } \phi_2^0 > 4 \\ \leq 0 & \text{otherwise} \end{cases} \)

\[ u_3 = H_1 \left( \frac{1}{2} \phi^{(3)} \right) = \begin{cases} \frac{1}{2} \phi^{(3)} = 4 - \phi_2^0 & \text{if } \phi_2^0 \\ \phi_2^0 = 6 - \phi_2^0 & \text{if } 6 \\ -\left( \sum_{j=1}^{n} \left( -f_j \phi^{(3)} \right) \right)^{-1} & \text{if } 4 < \phi_2^0 \leq 6 \end{cases} \]

\[ u_3 = \begin{cases} > 0 & \text{if } \phi_2^0 < 4 \text{ or } \phi_2^0 > 6 \\ \leq 0 & \text{otherwise} \end{cases} \]

\[ u_4 = \begin{cases} u_3 > 0 & \text{if } \phi_2^0 < 4 \text{ or } \phi_2^0 > 6 \\ -\left( \sum_{j=1}^{n} \left( -f_j \phi^{(4)} \right) \right)^{-1} & \text{otherwise} \end{cases} \]

\[ u_5 = \begin{cases} u_2 > 0 & \text{if } \phi_2^0 \leq 2 \text{ or } \phi_2^0 > 4 \\ -\left( \sum_{j=1}^{n} \left( -f_j \phi^{(5)} \right) \right)^{-1} & \text{otherwise} \end{cases} \]
\[ U(\Phi_2^0) = \begin{cases} \sum_{k \in K} U_k & \text{if } K \neq \text{null} \\ -\left(\frac{5}{\sum_{k=1}^{5} (-U_k)^{-1}}\right)^{-1} & \text{if } K = \text{null} \end{cases} \]

where \( K = \{ k \mid U_k > 0, \ k = 1, 2, 3, 4, 5 \} \).

It can be found that for any value of \( \Phi_2^0 \), there is always at least one \( U_k, k = 1, 2, 3, 4, 5 \), which is positive. Therefore index set \( K \) is never null.

\[
U(\Phi_2^0) = \sum_{k \in K} U_k = \begin{cases} U_1 + U_2 + U_3 + U_4 + U_5 & \text{if } \Phi_2^0 < 0, \ (K=\{1, 2, 3, 4, 5\}) \\ U_2 + U_3 + U_4 + U_5 & \text{if } 0 \leq \Phi_2^0 < 2, \ (K=\{2, 3, 4, 5\}) \\ U_1 + U_2 + U_3 & \text{if } 2 \leq \Phi_2^0 < 4, \ (K=\{1, 2, 3, 4\}) \\ U_1 + U_2 + U_4 & \text{if } 4 \leq \Phi_2^0 < 6, \ (K=\{1, 2, 5\}) \\ U_1 + U_2 + U_3 + U_4 + U_5 & \text{if } 6 \leq \Phi_2^0, \ (K=\{1, 2, 3, 4, 5\}) \end{cases}
\]

i.e.,

\[
U(\Phi_2^0) = \begin{cases} 12 - 5\Phi_2^0 & \text{if } \Phi_2^0 < 0 \\ 12 - 4\Phi_2^0 & \text{if } 0 \leq \Phi_2^0 < 2 \\ 6 - \Phi_2^0 & \text{if } 2 \leq \Phi_2^0 < 4 \\ 3\Phi_2^0 - 10 & \text{if } 4 \leq \Phi_2^0 < 6 \\ 5\Phi_2^0 - 22 & \text{if } 6 \leq \Phi_2^0 \end{cases}
\]
(2) From above figure, optimal value $\Phi_z^o = 4$

$\Phi_z^o = \min U(\Phi_z^o) = 2$.

(3) See Figure 1.

(4) See Figure 3.
Case 2

1. Case 2 can be solved in a way similar to case 1, i.e., first find

\[ u_k = \max_j \{ f_j (\Phi^k) \} \]

for \( k = 1, 2, 3, 4, 5 \) and then find

\[ u(\Phi^0) = \max_k \{ u_k \} \]

Here we present a different, simpler, (but not general) approach. (Graphic inspection approach)

\[ u(\Phi^0) = \max_k \{ u_k \} = \max_k \left\{ \max_j \{ f_j (\Phi^k) \} \right\} \]

\[ = \max_{j,k} \{ f_j (\Phi^k) \} \]

Notice that for this particular example,

\[ f_j (\Phi^k) = \text{distance from } \Phi^k \text{ to the} \]

boundary defined by \( f_j = 0 \),

see Figure 1

By inspecting Figure 1, either

\[ f_1 (\Phi^1) \]

or

\[ f_2 (\Phi^2) \]

will represent the most violation of specifications.
To be more specific,

\[ U(\phi_2^0) = \begin{cases} 
\phi_2^0 - 2 & \text{if } \phi_2^0 > 3 \\
4 - \phi_2^0 & \text{if } \phi_2^0 < 3
\end{cases} \]

(2) From Figure 4, optimal solution is \( \phi_2^0 = 3 \) and the minimum of \( U(\phi_2^0) \) is 1.

(3) See Figure 1.

(4) See Figure 5.
Comparison:

<table>
<thead>
<tr>
<th></th>
<th>Solution of Case 1</th>
<th>Solution of Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield (not counting φ°)</td>
<td>80%</td>
<td>40%</td>
</tr>
<tr>
<td>Worst Case Performance</td>
<td>largest violation of all specs by all the 5 circuits</td>
<td>1 = 2</td>
</tr>
</tbody>
</table>

Therefore case 2 is suitable for worst case design. Case 1 is suitable for yield maximization.