

- 1) $v_{o,n} = v_{o,n-1} - \alpha v_{in}$ for regular integrator with t_n at the end of ϕ_1
- 2) $v_{o,n} = v_{o,n-1} + \alpha v_{i,n-1/2}$ for negative resistor with t_n at the end of ϕ_2
- 3) $v_{o,n} = v_{o,n-1} + \alpha v_{i,n-1}$ for negative resistor with t_n at the end of ϕ_1

z transform

replace the subscripts $n-1$, $n-1/2$, and n with z^{-1} , $z^{-1/2}$ and z^{-0} multiplier. For example, the integrators from last section become:

$$1) \frac{v_o}{v_i} = -\frac{\alpha}{1-z^{-1}} \quad 2) \frac{v_o}{v_i} = \frac{\alpha z^{-1/2}}{1-z^{-1}} \quad 3) \frac{v_o}{v_i} = \frac{\alpha z^{-1}}{1-z^{-1}}$$

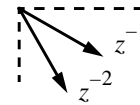
Here, z^{-1} represents a delay of 1 clock period T , where $z = e^{sT}$. The z transform is just a substitution for the Laplace transform for a delay. Here, the delay $T = 1/f_s$ is one period of the sampling clock.

$$z = e^{sT} = e^{j2\pi f T} = e^{j2\pi \frac{f}{f_s}} = 1 \angle 2\pi \frac{f}{f_s} = \cos\left(2\pi \frac{f}{f_s}\right) + j \sin\left(2\pi \frac{f}{f_s}\right)$$

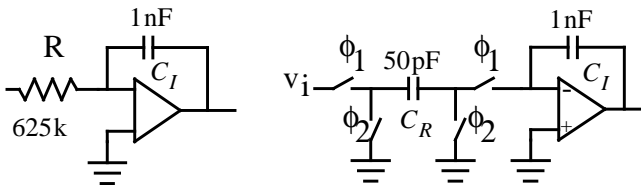
This is a vector of magnitude 1 and angle is f/f_s as a fraction of 360° . This results in the following table for frequency and values of z^{-1} .

f/f_s	0	1/8	1/4	1/2	1
vector direction	\rightarrow	\swarrow	\downarrow	\leftarrow	\rightarrow
z^{-1}	1 $= 1 \angle 0$	$1/\sqrt{2} - j1/\sqrt{2}$ $= 1 \angle -45^\circ$	$-j1$ $= 1 \angle -90^\circ$	-1 $= 1 \angle -180^\circ$	1 $= 1 \angle -360^\circ = 1 \angle 0$

note that $z^{-2} = e^{-j2 \cdot 2\pi \frac{f}{f_{clock}}}$, thus vector has twice the angle of z^{-1} .



Integrator Example (with positive resistor, clock frequency = 32 kHz, $C_R=50$ pF, $C_I=1$ nF)



$$R = \frac{T}{C_R} = \frac{1}{f_{clock} \cdot C_R} = \frac{1}{32\text{kHz} \cdot 50\text{pF}} = 625\text{k}\Omega$$

$$\text{time: } v_{o,n} - v_{o,n-1} = \alpha v_i, \text{ where } \alpha = \frac{C_R}{C_I} = 0.05$$


$$\text{frequency: } \frac{v_o}{v_i} = -\frac{\alpha}{1-z^{-1}}, \text{ where } z^{-1} = e^{-j2\pi \frac{f}{f_{clock}}}$$

$$\text{Compare to continuous: time: } \frac{dv_o}{dt} = -\frac{v_i}{RC_I}, \text{ frequency: } \frac{v_o}{v_i} = -\frac{1}{sRC_I} \Rightarrow -\frac{1}{j\omega RC_I} \Rightarrow -\frac{\alpha f_{clock}}{j\omega}$$

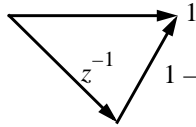
$$\omega_u = \frac{1}{RC_I} = \alpha f_{clock}, \text{ (using } R = \frac{T}{C_R}, \alpha = \frac{C_R}{C_I}), \text{ then } f_u = 254.6 \text{ Hz.}$$

Frequency Response Analysis of S.C. Implementation

At DC (or close to it) $f \approx 0$, $z^{-1} \approx 1$, $\frac{v_o}{v_i} \approx -\frac{\alpha}{1-1} \Rightarrow \infty \angle 90^\circ$ (not to scale)

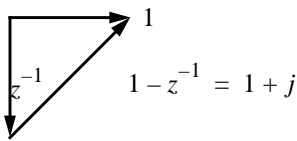


at $f = \frac{f_{clock}}{8}$, $1 - z^{-1} = 0.293 + j0.707$, $\frac{v_o}{v_i} = -\frac{0.05}{0.293 + j0.707}$



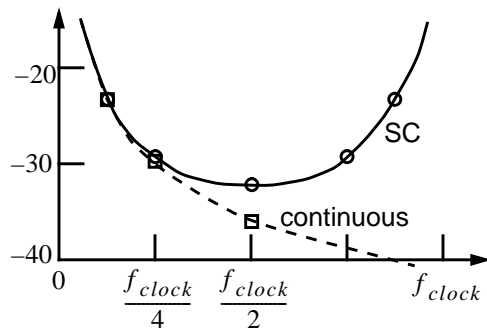
$\frac{v_o}{v_i} = -0.025 + j0.06 = 0.065 \angle 67.5^\circ$ Gain = -23.7 dB (continuous = -23.9 dB)

at $f = \frac{f_{clock}}{4}$, $1 - z^{-1} = 1 + j$, $\frac{v_o}{v_i} = -\frac{0.05}{1 + j} = 0.035 \angle 45^\circ$, Gain = -29 dB (RC=-29.9 dB)



at $f = \frac{f_{clock}}{2}$, $1 - z^{-1} = 1 + 1 = 2$, $\frac{v_o}{v_i} = -\frac{0.05}{2} = -0.025$, Gain = -32 dB (Continuous RC=-36 dB)

at $f = f_{clock}$, $z^{-1} = 1$, $\frac{v_o}{v_i} = -\frac{\alpha}{1-1}$ same as DC.



Good agreement at low frequencies. Can show mathematically:

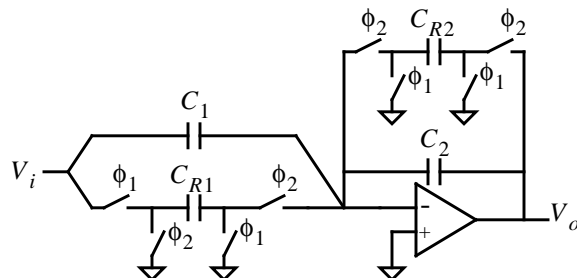
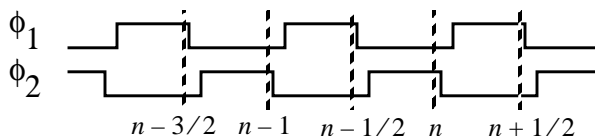
Since $z^{-1} = e^{-j2\pi \frac{f}{f_{clock}}}$, for small x , $e^x \approx 1 + x \dots$, or
 $z^{-1} \approx 1 - j2\pi \frac{f}{f_{clock}}$, or $1 - z^{-1} \approx j2\pi \frac{f}{f_{clock}} = j\omega T = j\omega R\alpha C_I$

Thus $\frac{-\alpha}{1 - z^{-1}} \approx -\frac{1}{j\omega R\alpha C_I}$ or $-\frac{\alpha f_{clock}}{j\omega}$ same as continuous

Writing Difference Equations and an Example (by R.D.B years ago)

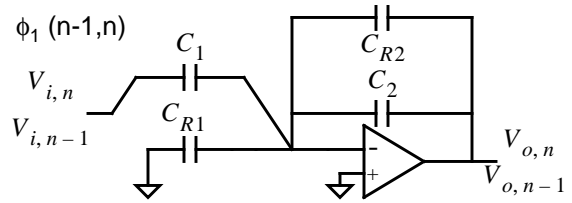
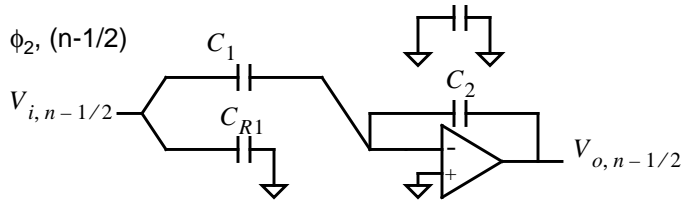
The basic objective is to determine the relationship between changes in input voltage and changes in output voltage, as they occur over *one full clock period*. In general, when two-phase non-overlapping clocks are used, this must be accomplished in two steps, e.g: from ϕ_2 to ϕ_1 and then from ϕ_1 to ϕ_2 which is one full period (ϕ_2 to ϕ_2).

For clocks as defined below, one must determine the difference equation from $t = (n-1)T$ to $t = nT$, using $t = (n-1/2)T$ as an intermediate step.



Procedure (Charge balance method) e.g., for circuit above.

1. Re-draw the circuit during the two clock phases. Note that the circuit for ϕ_2 will be used twice, once for $(n-1)$ and once for n . The circuit for ϕ_1 is used only for $(n-1/2)$.



2. Using your diagrams as aids, and for each of the two transitions $(n-1) \Rightarrow (n-1/2)$ and $(n-1/2) \Rightarrow (n)$, write Kirchoff's current equation at the inverting input to the opamp including currents through all capacitors that are connected directly to the inverting input at the "instant in question."

Kirchoff's current law for switched capacitors is as follows:

$$\sum i = 0 \Rightarrow \sum \frac{\Delta Q}{\Delta t} = 0 \Rightarrow \sum \frac{C \Delta V}{T} = 0$$

where $V = V_A - V_B$, and $\Delta V = V_{final} - V_{initial} = V_{now} - V_{half\ period\ ago}$

3. Solve the two equations simultaneously to eliminate $v_{o,n-1/2}$

For the example: a) for $(n-1) \Rightarrow (n-1/2)$, the "instant in question" is $(n-1/2)$

$$C_2 \Delta V_{C_2} + C_1 \Delta V_{C_1} = 0 \quad \text{or} \quad C_2 [V_{C_2, n-1/2} - V_{C_2, n-1}] + C_1 [V_{C_1, n-1/2} - V_{C_1, n-1}] = 0.$$

$$C_2 [(V_{o, n-1/2} - 0) - (V_{o, n-1} - 0)] + C_1 [(V_{i, n-1/2} - 0) - (V_{i, n-1} - 0)] = 0$$

note: all the zeros are from the inverting input = virtual ground. Removing these results in:

$$C_2 (V_{o, n-1/2} - V_{o, n-1}) + C_1 (V_{i, n-1/2} - V_{i, n-1}) = 0$$

similarly, b) for $(n-1/2) \Rightarrow (n)$ the "instant in question" is now n .

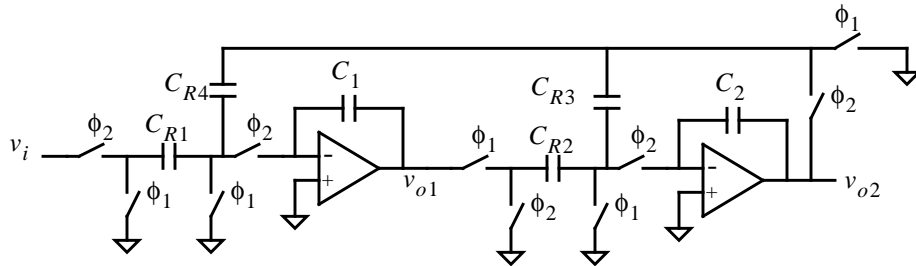
$$C_2 [(V_{o, n} - 0) - (V_{o, n-1/2} - 0)] + C_{R_2} [(V_{o, n} - 0) - (0 - 0)] +$$

$$C_1 [(V_{i, n} - 0) - (V_{i, n-1/2} - 0)] + C_{R_1} [(0 - 0) - (V_{i, n-1/2} - 0)] = 0$$

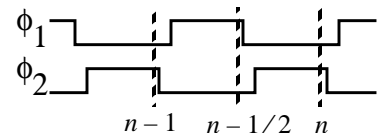
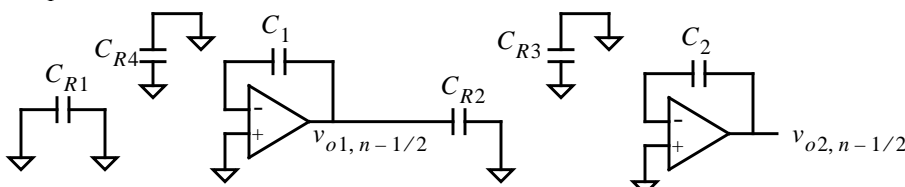
Removing all the zeros:

$$C_2 [V_{o, n} - V_{o, n-1/2}] + C_{R_2} [V_{o, n}] + C_1 [V_{i, n} - V_{i, n-1/2}] - C_{R_1} [V_{i, n-1/2}] = 0$$

4. Note 1: Sometimes the only equation you can get for a transition is for example: $V_{o, n-1/2} = V_{o, n-1}$
Note 2: If multiple opamps, then repeat above for each opamp.

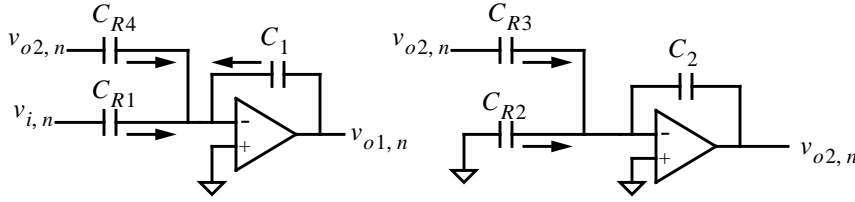
Example with Biquad Filter

On ϕ_1



Equations: $V_{o1, n-1/2} = V_{o1, n-1}$, $V_{o2, n-1/2} = V_{o2, n-1}$

On ϕ_2



Equation1: $C_{R1}v_{i,n} + C_{R4}v_{o2,n} + C_1(v_{o1,n} - v_{o1,n-1/2}) = 0$, divide by C_1 , substitute ϕ_1 equations and take a z

transform, $\alpha_1 v_i + \alpha_4 v_{o2} + v_{o1} - v_{o1}z^{-1} = 0$ (1) or $v_{o1} = -\frac{\alpha_1 v_i + \alpha_4 v_{o2}}{1 - z^{-1}}$ (2)

equation 2: $C_{R2}(0 - v_{i,n-1/2}) + C_2(v_{o2,n} - v_{o2,n-1/2}) + C_{R3}(v_{o2,n} - 0) = 0$ (3) divide by C_2 , substitute ϕ_1 equa-

tion, z transform, $-\alpha_2 v_{o1}z^{-1} + v_{o2} - v_{o2}z^{-1} + \alpha_3 v_{o2} = 0$ (4) or collecting terms, $v_{o2}(1 + \alpha_3 - z^{-1}) = \alpha_2 v_{o1}z^{-1}$ (5)

substituting (2) into (5) $v_{o2}(1 + \alpha_3 - z^{-1}) + \alpha_2 \frac{\alpha_1 v_i + \alpha_4 v_{o2}}{1 - z^{-1}} z^{-1} = 0$ or

$$\frac{v_{o2}}{v_i} = \frac{-\frac{\alpha_2 \alpha_1 z^{-1}}{1 - z^{-1}}}{\frac{\alpha_2 \alpha_4 z^{-1}}{1 - z^{-1}} + 1 + \alpha_3 - z^{-1}} = \frac{-\alpha_1 \alpha_2 z^{-1}}{(1 + \alpha_3 - z^{-1})(1 - z^{-1}) + \alpha_2 \alpha_4 z^{-1}} = \frac{-\alpha_1 \alpha_2 z^{-1}}{z^{-2} - z^{-1}(2 + \alpha_3 - \alpha_2 \alpha_4) + 1 + \alpha_3}$$

Biquad Equation by Integrator Analogy

By making use of the two equations: $\Delta v_o = -\alpha v_{i,n}$ for a normal integrator with regular resistor and $\Delta v_o = \alpha v_{i,n-1/2}$, or $\Delta v_o = \alpha v_{i,n-1}$ for an inverting resistor, we can write equations for the two integrators.

equation 1: $v_{o1,n} = v_{o1,n-1} - \alpha_1 v_{i,n} - \alpha_4 v_{o2,n}$ apply z transform: $v_{o1} = v_{o1}z^{-1} - \alpha_1 v_i - \alpha_4 v_{o2}$ or

$$v_{o1} = \frac{-\alpha_1 v_i - \alpha_4 v_{o2}}{1 - z^{-1}}$$

equation 2: $v_{o2,n} = v_{o2,n-1} + \alpha_2 v_{o1,n-1} - \alpha_3 v_{o2,n}$ apply z transform: $v_{o2} = v_{o2}z^{-1} + \alpha_2 v_{o1}z^{-1} - \alpha_3 v_{o2}$ or

$v_{o2}(1 + \alpha_3 - z^{-1}) = \alpha_2 z^{-1} v_{o1}$. Now substitute for v_{o1} from above, result is:

$$v_{o2}(1 + \alpha_3 - z^{-1}) = \alpha_2 z^{-1} \left[\frac{-\alpha_1 v_i - \alpha_4 v_{o2}}{1 - z^{-1}} \right], \text{ separate out } v_{o2} \text{ and } v_{o1} \text{ with the result:}$$

$$v_{o2}[(1 + \alpha_3 - z^{-1})(1 - z^{-1}) + \alpha_2 \alpha_4 z^{-1}] = -\alpha_1 \alpha_2 z^{-1} v_i \quad \text{or} \quad \frac{v_{o2}}{v_i} = \frac{-\alpha_1 \alpha_2 z^{-1}}{z^{-2} - z^{-1}(2 + \alpha_3 - \alpha_2 \alpha_4) + 1 + \alpha_3}$$

Time Domain Analysis of Biquad

The frequency domain equation is of the form: $\frac{v_{o2}}{v_i} = -\frac{cz^{-1}}{z^{-2} - az^{-1} + b}$ where a , b and c are constants calculated

from capacitor ratios. Cross multiply to result in: $v_{o2}(z^{-2} - az^{-1} + b) = -v_i cz^{-1}$, taking the inverse z transform:

$$v_{o2,n-2} - av_{o2,n-1} + bv_{o2} = -cv_{i,n-1}, \quad \text{or} \quad v_{o2} = \frac{a}{b}(v_{o2,n-1}) - \frac{1}{b}(v_{o2,n-2}) - \frac{c}{b}(v_{i,n-1})$$

Thus start with $v_{i,n} = v_{i,n-1} = v_{o2,n} = v_{o2,n-1} = v_{o2,n-2} = 0$, then change $v_{i,n} = 1$ and calculate new $v_{o2,n}$.